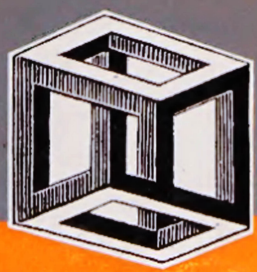
D. S. Mitrinović

Differential Geometry



Tutorial Texts

TUTORIAL TEXT NO. 5

DIFFERENTIAL GEOMETRY

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FOREWORD

The problems contained in this series have been collected over many years with the aim of providing students and teachers with material, the search for which would otherwise occupy much valuable time. Hitherto, this concentrated material has only been accessible to the very restricted public able to read Serbian.

I greatly welcome the appearance of the Tutorial Texts based on my problem collection. Cooperation with colleagues in Australia and the U.S.A. and their initiative have made it possible. For the preparation of this Text, I wish to thank R. S. Anderssen.

I am sorry that this book will not be seen by my co-author, J. Ulčar, who died at the beginning of 1967, and who is responsible for the greater part of the book.

Belgrade, February 1969 D. S. MITRINOVIĆ

TABLE OF CONTENTS

Page

CHAPTER 0 Introduction 7

Part 1: Two-Dimensional Differential Geometry

CHAPTER 1 Elements of two-dimensional differential geometry 9

1.1 The concept of curve 9
1.2 Arc length 10
1.3 The tangent and the normal 11
1.4 Curvature 12
1.5 The intrinsic equation of a curve 13
1.6 The circle of curvature and the evolute . . . 13
1.7 The involute 14
1.8 The envelope 15

CHAPTER 2 Problems on two-dimensional differential geometry 16

2.1 Forms and curves 16
2.2 Length 18
2.3 Tangent and normal 21
2.4 Curvature 27
2.5 The intrinsic equation 30
2.6 Circles of curvature and evolutes 34
2.7 Involutes 38
2.8 Envelopes 39

Part II: Three-Dimensional Differential Geometry

CHAPTER 3 Elements of the theory of space curves 43

3.1 The concept of curve 43
3.2 The unit tangent 43
3.3 The arc length 43
3.4 The osculating plane 44

3.5 The tangent, normal and binormal 44
3.6 The Serret-Frenet formulae 45
3.7 The spherical indicatrix. 46
3.8 The circle of curvature and the osculating sphere 46

CHAPTER 4 Problems on three-dimensional space curves . . . 48

CHAPTER 5 Elements of the theory of surfaces 61

5.1 The concept of surface 61
5.2 The tangent plane and the normal 61
5.3 The envelope 63
5.4 Developable surfaces. 63
5.5 The involutes and evolutes of a curve. . . . 64
5.6 Length. 64
5.7 Angle 65
5.8 The second fundamental form of a surface and the curvature of a surface curve 66
5.9 The theorem of Meusnier. The normal section 67
5.10 The indicatrix of Dupin and the formula of Euler 67
5.11 The Gaussian and average curvature 68
5.12 The theorem of Gauss and Christoffel symbols 68
5.13 The conjugate directions and asymptotic lines 69
5.14 Lines of curvature 69
5.15 Geodesic curvature and geodesics 70
5.16 The mapping of surfaces 71

CHAPTER 6 Problems relating to the properties of surfaces . . 73

6.1 Surfaces 73
6.2 The tangent plane and the normal 75
6.3 Envelope and edge of regression 79
6.4 Developable and ruled surfaces 80
6.5 Involutes and evolutes 83
6.6 Length and the first fundamental form . . . 84

6.7 Angles, nets and trajectories 86
6.8 The second fundamental form and curvature. 92
6.9 Normal curvature 93
6.10 Principal curvatures and directions. Umbilical points 94
6.11 Gaussian and average curvature 96
6.12 The theorem of Gauss. Christoffel symbols . 100
6.13 Conjugate directions and asymptotic lines . . 101
6.14 Lines of curvature 107
6.15 Geodesic curvature and geodesics 109
6.16 The mapping of surfaces 112

References 120

INTRODUCTION

0.1 This book consists of a collection of problems on elementary two- and three-dimensional differential geometry with a brief discussion of the relevant theory. The book is not intended as an independent treatise on elementary differential geometry; but should be regarded as a problem supplement to some standard text. Such texts include those listed as references at the end of this book. The reader is referred to these for further clarification and discussion of topics treated in this book.

The book is divided into two parts. Part I deals with the two-dimensional theory and problems, while Part II is more comprehensive and deals with the three-dimensional theory and problems. Part I is introductory and for this reason consists basically of a collection of instructive problems with little comment. More attention is paid to the solution of problems in Part II.

The *elements* of the theory are found in the odd numbered chapters, while the related collections of *problems* appear in the corresponding sections of the succeeding even numbered chapters.

0.2 The study of differential geometry requires of the reader a certain mathematical foundation.

0.2.1. The reader should be completely familiar with the methods of differentiation and integration including the integration of second order linear ordinary differential equations as found, for example, in H. Lamb, *An elementary course of infinitesimal calculus*, Cambridge University Press, 1956.

0.2.2. A knowledge of the elements of vector analysis is also required. In particular, a knowledge of the differentiation and integration of vectors and the manipulation of the scalar and vector products of two vectors (see, for example, C. E. Weatherburn, *Elementary vector analysis with applications to geometry and physics*, Bell, London, 1921). The $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ notation for Cartesian coordinates in three-dimensions is used.

0.2.3. It is hoped that by using the g_{ik} and the k_{ik} symbols as well as the Christoffel symbols further insight into the use and the manipulation of tensor notation will be gained by the student.

0.3. Occasionally, reference will be made to a publication listed as a reference at the end of the book by using the corresponding number in square brackets (for example, [8] refers to the book by T. J. Willmore).

Part I Two-Dimensional Differential Geometry

CHAPTER 1

ELEMENTS OF TWO-DIMENSIONAL DIFFERENTIAL GEOMETRY

1.1 THE CONCEPT OF CURVE

The point set $\boldsymbol{r}=\{x(t), y(t)\}$, $a \leqslant t \leqslant b$, defines a *smooth arc* of a curve when the coordinate functions $x(t)$, $y(t)$ are continuous on their interval of definition $a \leqslant t \leqslant b$, and when the first derivatives of $x(t)$ and $y(t)$ exist, are continuous, and are non-zero on $a \leqslant t \leqslant b$.

The equations

$$x = x(t), \quad y = y(t), \qquad a \leqslant t \leqslant b\,, \tag{1.1.1}$$

where $x(t)$ and $y(t)$ define the coordinates of the position vector which determines the smooth arc of a curve, are called the *parametric representation* or *parametric form* of this arc, while

$$\boldsymbol{r} = \boldsymbol{r}(t) = \{x(t), y(t)\}\,, \qquad a \leqslant t \leqslant b\,,$$

is called the *vector form* of this arc.

If t is eliminated from (1.1.1), then the *implicit form*

$$F(x, y) = 0 \tag{1.1.2}$$

is obtained. The points satisfying (1.1.2) form a curve which includes all the points of the smooth arc (1.1.1).

Since for a smooth arc, $\dot{x} = \mathrm{d}x(t)/\mathrm{d}t \neq 0$, $a \leqslant t \leqslant b$, the inverse function $t = t(x)$ exists in the interval $x(a) \leqslant x \leqslant x(b)$, $\dot{x} > 0$, or $x(b) \leqslant x \leqslant x(a)$, $\dot{x} < 0$. If $t = t(x)$ is substituted in $y = y(t)$, then $y = y\{t(x)\} = f(x)$. Hence in the interval $a \leqslant t \leqslant b$, the smooth arc (1.1.1) can be expressed in the *explicit form*

$$y = f(x)\,. \tag{1.1.3}$$

Polar coordinates r, θ are often used instead of Cartesian coordinates. If

$$x = r\cos\theta,\ y = r\sin\theta$$

is substituted in (1.1.2), the *polar form*

$$\Theta(r, \theta) = 0$$

is obtained.

NOTE. The smooth arc (1.1.1) will often be referred to as the curve (1.1.1) or the curve $\boldsymbol{r}(t)$.

1.2 ARC LENGTH

The point of (1.1.1) which corresponds to the value $t=\tau$ has coordinates $(x(\tau), y(\tau))$ and is often called the point τ of the curve. The *arc length* s between the points a and b of the curve (1.1.1) is usually defined by the formula

$$s = \int_a^b (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}\,\mathrm{d}t\,, \tag{1.2.1}$$

which, in vector form, becomes

$$s = \int_a^b (\dot{\boldsymbol{r}}^2)^{\frac{1}{2}}\,\mathrm{d}t\,, \qquad \dot{\boldsymbol{r}}^2 = \dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}\,.$$

NOTE. The notation $\boldsymbol{r}^2 = (\boldsymbol{r})^2 = \boldsymbol{r}\cdot\boldsymbol{r}$ will be used throughout this book.

When the curve (1.1.1) is such that the above arc length exists, the curve is said to be *rectifiable*. Under the above assumptions on (1.1.1), this will always be the case.

For the function

$$s = s(t) = \int_a^t (\dot{\boldsymbol{r}}^2)^{\frac{1}{2}}\,\mathrm{d}t\,,$$

the inverse function $t=t(s)$ exists. Substitution in $\boldsymbol{r}(t)$ leads to $\boldsymbol{r} = \boldsymbol{r}(t(s)) = \boldsymbol{r}(s)$. Hence, the parameter t of a curve can be replaced by the arc length s.

If the equation of a curve is given in the form (1.1.3), formula (1.2.1) reduces to

$$s = \int_a^b \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{1}{2}}\mathrm{d}x\,.$$

1.3 THE TANGENT AND THE NORMAL

The vector $\dot{\boldsymbol{r}}(t)$ is called the *tangent vector* to $\boldsymbol{r}(t)$ at t. The equation of the *tangent* to $\boldsymbol{r}(t)$ at t is

$$\boldsymbol{R} = \boldsymbol{r}(t) + \lambda \dot{\boldsymbol{r}}(t), \tag{1.3.1}$$

where λ is the variable which defines the points on the tangent. In parametric form, this becomes

$$\begin{aligned} X &= x(t) + \lambda \dot{x}(t), \\ Y &= y(t) + \lambda \dot{y}(t). \end{aligned} \tag{1.3.2}$$

Elimination of λ from (1.3.2) gives the equation of the tangent in the form

$$\begin{vmatrix} X - x(t) & Y - y(t) \\ \dot{x}(t) & \dot{y}(t) \end{vmatrix} = 0. \tag{1.3.3}$$

For all curves of the form $y = f(x)$, the equation of the tangent at (x_0, y_0) reduces to

$$y - y_0 = y_0'(x - x_0),$$

where $y_0' = \mathrm{d}f(x_0)/\mathrm{d}x$.

A vector which is normal to the tangent vector $\dot{\boldsymbol{r}}(t)$ at the point t is called the *normal vector* to $\boldsymbol{r}(t)$ at t. Since the scalar product of the two vecotrs $\dot{\boldsymbol{r}} = \{\dot{x}, \dot{y}\}$ and $\{\dot{y}, -\dot{x}\}$ is zero, it follows that the vectors $\pm\{\dot{y}, -\dot{x}\}$ are normal vectors to $\dot{\boldsymbol{r}}$. Thus, the *normal* to the curve $\boldsymbol{r}(t)$ at t has the parametric form

$$\begin{aligned} X &= x(t) + \lambda \dot{y}(t), \\ Y &= y(t) - \lambda \dot{x}(t), \end{aligned} \tag{1.3.4}$$

where λ is the variable parameter.

For all curves of the form $y = f(x)$, the parametric form (1.3.4) of the normal reduces to

$$y - y_0 = -\frac{1}{y_0'}(x - x_0),$$

where $y_0' = \mathrm{d}f(x_0)/\mathrm{d}x$.

1.4 CURVATURE

Consider the curve $\boldsymbol{r}=\boldsymbol{r}(s)$, where s is its arc length. If the curve is not straight, it is important to have a measure of its deviation from a straight line. Such a measure, called *the curvature* $\kappa(s)$, is given by the rate at which the tangent changes direction as it moves along the curve. Let $\phi(s)$ be the angle, measured in the positive sense, between the tangent direction $\boldsymbol{r}'(s)$ at s and the direction $\boldsymbol{i}$. The curvature $\kappa(s)$ of a curve $\boldsymbol{r}(s)$ at the point s is then defined by

$$\kappa(s) = \frac{\mathrm{d}\phi(s)}{\mathrm{d}s}. \tag{1.4.1}$$

It follows from (1.4.1) that

$$\kappa = x'y'' - y'x''. \tag{1.4.2}$$

The vector $\boldsymbol{r}''$ is a vector normal to the tangent vector $\boldsymbol{r}'$ such that

$$|\kappa| = |\boldsymbol{r}''|. \tag{1.4.3}$$

If $\kappa \neq 0$, it follows that

$$\boldsymbol{t} = \boldsymbol{r}', \quad \boldsymbol{n} = \frac{1}{\kappa}\boldsymbol{r}'', \tag{1.4.4}$$

where $\boldsymbol{t}$ and $\boldsymbol{n}$ are the *unit tangent* and *unit normal* to the curve $\boldsymbol{r}(s)$ at s. At every point of a curve, the vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ are equally inclined to $\boldsymbol{i}$ and $\boldsymbol{j}$, respectively.

The sign of the curvature, which has a geometrical interpretation, is given by (1.4.2). If $\boldsymbol{r}''$ is continuous at the point s_0, then, in a neighbourhood of the point s_0 of $\boldsymbol{r}=\boldsymbol{r}(s)$, the curve lies on the side of the tangent to which $\boldsymbol{n}$ points when $\kappa(s_0) > 0$ and on the opposite side when $\kappa(s_0) < 0$.

As a rule, a curve will be given in parametric form where the parameter t is not the arc length s. In this case the curvature is given by

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}. \tag{1.4.5}$$

For all curves of the form $y=f(x)$, (1.4.5) reduces to

$$\kappa = \frac{y''}{(1+y'^2)^{\frac{3}{2}}}, \quad y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}.$$

The *radius of curvature* ρ of $\boldsymbol{r}(t)$ at t_0 is given by

$$\rho(t_0) = 1/\kappa(t_0) \qquad (\kappa(t_0) \neq 0).$$

1.5 THE INTRINSIC EQUATION OF A CURVE

Using the fact that $\boldsymbol{t}$ and $\boldsymbol{n}$ are orthogonal unit vectors, it follows from (1.4.4) that

$$\boldsymbol{t}' = \kappa \boldsymbol{n}, \qquad \boldsymbol{n}' = -\kappa \boldsymbol{t}\ ; \tag{1.5.1}$$

these are the so called *Frenet formulae* for plane curves. Therefore, there exists for every curve $\boldsymbol{r}(s)$ a function $\kappa(s)$ which satisfies the differential equations (1.5.1).

The converse is also true: If $\kappa(s)$, $a \leqslant s \leqslant b$, is an arbitrary continuous function, then there exists a twice continuously differentiable smooth curve $\boldsymbol{r} = \boldsymbol{r}(s)$, $a \leqslant s \leqslant b$, the curvature of which is equal to $\kappa(s)$, where s is the arc length of the curve.

The curve, which has curvature $\kappa(s)$ at the point s, has the parametric form

$$x = \int_a^t \cos\phi \, ds, \qquad y = \int_b^t \sin\phi \, ds,$$

where $\phi = \int \kappa(s) ds$ and (a, b) is a fixed point on the curve.

The curves $\boldsymbol{r} = \boldsymbol{r}(s)$ and $\kappa = \kappa(s)$ are congruent; the equation $\boldsymbol{r} = \boldsymbol{r}(s)$ defines a curve with a fixed position in the plane, whereas $\kappa = \kappa(s)$ defines a congruent curve with no fixed position in the plane. For this reason, the equation $\kappa = \kappa(s)$ is called the *intrinsic equation* of the curve $\boldsymbol{r}(s)$.

1.6 THE CIRCLE OF CURVATURE AND THE EVOLUTE

Let the point t_0 of the curve C, $\boldsymbol{r} = \boldsymbol{r}(t)$, be such that $\kappa(t_0) \neq 0$. The position vector (see **1.4**)

$$\boldsymbol{R} = \boldsymbol{r}(t_0) + \rho(t_0)\boldsymbol{n}(t_0) \tag{1.6.1}$$

defines the *centre of curvature* of C at t_0. The radius and centre of

curvature of C at t_0 determine the *circle of curvature* of C at t_0.

Let $\boldsymbol{R} = \boldsymbol{R}(t) = \{X(t), Y(t)\}$ and write X, Y, x and y instead of $X(t)$, $Y(t)$, $x(t)$ and $y(t)$.

Since

$$\begin{aligned}\boldsymbol{n} &= \{-\dot{y}(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}, \dot{x}(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}\} \\ &= \{-\dot{y}\dot{s}^{-1}, \dot{x}\dot{s}^{-1}\},\end{aligned}$$

equation (1.6.1) becomes

$$X = x - \rho\frac{\dot{y}}{\dot{s}}, \quad Y = y + \rho\frac{\dot{x}}{\dot{s}}, \tag{1.6.2}$$

or, using (1.4.5),

$$X = x - \frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}\dot{y}, \quad Y = y + \frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}\dot{x}. \tag{1.6.3}$$

For all curves of the form $y=f(x)$, formula (1.6.3) becomes

$$X = x - \frac{1+y'^2}{y''}y', \quad Y = y + \frac{1+y'^2}{y''}.$$

The geometric locus of the centre of curvature of a curve C is called its *evolute*. The normal to a given curve C is tangent to its evolute.

1.7 THE INVOLUTE

The tangent to the evolute E of a curve C cuts C at right angles. Every curve which the tangent to E cuts at right angles is an *involute* of E. This can be illustrated simply: Imagine a string stretched tightly around a given curve E. The end P of the string moves in the plane of E so that the string remains taut. As the string is unwound, the point P traces out an involute of E.

If $\boldsymbol{r} = \boldsymbol{r}(s)$ is the equation of the curve E and s is its arc length, then the equation of its involutes is given by

$$\boldsymbol{R} = \boldsymbol{r} + (C_0 - s)\,\boldsymbol{r}',$$

where C_0 is the parameter with respect to which the involutes of E are determined.

The parametric form of this equation is

$$X = x(s) + (C_0 - s)x'(s),$$
$$Y = y(s) + (C_0 - s)y'(s).$$

1.8 THE ENVELOPE

Consider the curves

$$\boldsymbol{r} = \boldsymbol{r}(t, \lambda), \tag{1.8.1}$$

defined for two parameter t and λ. Let $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$ and $\partial \boldsymbol{r}/\partial\lambda = \boldsymbol{r}_\lambda$ be continuous for a range of values of t and λ and assume that $\dot{\boldsymbol{r}} \neq 0$ for these values. For every fixed value of λ, equation (1.8.1) defines a curve with respect to the variable t. Equation (1.8.1) is said to define a one-parameter family of curves, where λ is the parameter of the family—called the *family parameter*. The curve which is not contained in the family, but which is tangent at every point to a member of the family, is called the *envelope* of the family.

It can be shown that the envelope of (1.8.1) consists of those points at which the determinant of the components of $\dot{\boldsymbol{r}}$ and $\boldsymbol{r}_\lambda$ vanishes, that is, for which

$$|\dot{\boldsymbol{r}} \times \boldsymbol{r}_\lambda| = 0, \tag{1.8.2}$$

if equations (1.8.1) and (1.8.2) together do not determine a curve $\lambda = \text{const.}$ of the family (1.8.1).

If, for example, equation (1.8.2) is satisfied for $\lambda = \lambda(t)$, then

$$\boldsymbol{r} = \boldsymbol{r}\{t, \lambda(t)\}$$

determines the envelope of the family (1.8.1).

If a family of curves is not given in vector form, but is defined by the equation

$$f(x, y, \lambda) = 0, \tag{1.8.3}$$

where λ is the family parameter, and if f is such that $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ do not both hold at the same time, then the envelope of the family (1.8.3) is defined by

$$f(x, y, \lambda) = 0, \quad \frac{\partial}{\partial \lambda}\{f(x, y, \lambda)\} = 0, \tag{1.8.4}$$

if these equations do not determine a curve $\lambda = \text{const.}$

Elimination of λ in (1.8.4) leads to the required envelope in the form $F(x, y) = 0$.

CHAPTER 2

PROBLEMS ON TWO-DIMENSIONAL DIFFERENTIAL GEOMETRY

2.1 FORMS AND CURVES

2.1.1 Find and sketch the curves with the parametric forms

(a) $x=t,\ y=t^2$;
(b) $x=t^2,\ y=t^4$;
(c) $x=a\cos t,\ y=b\sin t$;
(d) $x=\cos t,\ y=\sin^2 t$;
(e) $x=e^t,\ y=e^{-t}$;
(f) $x=\sqrt{1-t},\ y=1/t.$

ANSWERS:

(a) $y=x^2$;
(b) $y=x^2,\ x\geqslant 0$;
(c) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$;
(d) $y=1-x^2,\ -1\leqslant x\leqslant 1$;
(e) $xy=1,\ x>0$;
(f) $y=(1-x^2)^{-1},\ x\neq 1.$

2.1.2 Discuss the curves defined by

(a) $x=2+3\sin t,\ y=-1+\sin t$;
(b) $x=t,\ y=\sqrt{t}$;
(c) $x=1+e^t,\ y=5+2e^t$;
(d) $x=a\cot t,\ y=\dfrac{a}{\sin^2 t}$;
(e) $x=t^2-t+1,\ y=t^2+t+1$;
(f) $x=t^2-2t+3,\ y=t^2-2t+1$;
(g) $x=a\cos^2 t,\ y=b\sin^2 t$;
(h) $x=a\cosh t,\ y=b\sinh t$;

(i) $x=a\cosh^2 t,\ y=b\sinh^2 t$;

(j) $x=(\arcsin\sqrt{1-t^2})^{-1},\ y^2=1-t^2$;

(k) $x=\tan t,\ y=\operatorname{cosec} t\sec t$.

2.1.3 Rewrite in an implicit or explicit form

(a) $x=\cos 2t,\ y=\tan t$;

(b) $x=\sin t,\ y=\cos\frac{t}{2}$;

(c) $x=\frac{1+\cos t}{\sin t},\ y=\tan t$;

(d) $x=\sin 2t,\ y=\tan 2t\ \cos t$;

(e) $x=\frac{1}{t(t-1)},\ y=\frac{2t}{t^2-1}$.

2.1.4 The *epicycloid* is the path of a point on the circumference of a circle K' which rolls on the outside of a circle K. Find the equation of the epicycloid when the radii of the circles K and K' are R and r, where $R>r$.

ANSWERS:

$$x=(R+r)\cos t-r\cos\frac{R+r}{r}t,$$

$$y=(R+r)\sin t-r\sin\frac{R+r}{r}t.$$

2.1.5 The *hypocycloid* is the path of a point on the circumference of a circle K' which rolls on the inside of a circle K. Determine the equation of the hypocycloid when the radii of the circles K and K' are R and r, where $R>r$.

ANSWERS:

$$x=(R-r)\cos t+r\cos\frac{R-r}{r}t,$$

$$y=(R-r)\sin t-r\sin\frac{R-r}{r}t.$$

2.1.6 Prove that if a circle K' rolls inside a circle K with twice its radius, then every point on the circumference of K' describes a diameter of K and every other point of K' describes an ellipse.

ANSWER:

$$x = (r+d)\cos t, \quad y = (r-d)\sin t,$$

where $2r$ is the radius of K and d denotes the distance of the point defining the locus from the centre of K'.

2.1.7 The path of a point on the circumference of a circle, which rolls on a straight line, is called a *cycloid*. Find the equation of the cycloid.

2.1.8 Sketch the following curves (where a and k are constants):

(a) the logarithmic spiral $r = ke^{a\theta}$;
(b) the Archimedian spiral $r = a\theta + k$;
(c) the hyperbolic spiral $r\theta = \pi$;
(d) the Fermat spiral $r^2 = a^2\theta$;
(e) the four-leaved rose $r = a\cos 2\theta$;
(f) the three-leaved rose $r = \cos 3\theta$;
(g) the leminiscate $r^2 = a^2 \cos 2\theta$.

2.1.9 Rewrite the equation of the curve

$$x = e^{-t}(\cos t + \sin t), \quad y = e^{-t}(\cos t - \sin t),$$

in polar form.

ANSWER:

$$r = \sqrt{2}\, e^{\theta - \pi/4} .$$

2.1.10 Rewrite in polar form

(a) $x = \tan t, \quad y = \sin t \cos t$;
(b) $(x^2+y^2)^2 = x^3 - 3xy^2$;
(c) $y^2 = 4x$.

2.2 LENGTH

2.2.1 Determine the length of the curves

(a) $y = a \cosh\left(\frac{x}{a}\right), \quad 0 \leqslant x \leqslant b$;

(b) $y=\log x, \quad \sqrt{3} \leqslant x \leqslant \sqrt{8}$;

(c) $y=a \log \cos\left(\frac{x}{a}\right), \quad 0 \leqslant x \leqslant b$;

(d) $y^3=px^2, \quad 0 \leqslant y \leqslant a$;

(e) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (the astroid) ;

(f) $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$;

(g) $r=a\,e^{m\theta}$ (the logarithmic spiral), $0 \leqslant r \leqslant a$;

(h) $r=a\theta$ (the Archimedes spiral), $0 \leqslant \theta \leqslant 2\pi$.

ANSWERS:

(a) $a \sinh\left(\frac{b}{a}\right)$;

(b) $1+\frac{1}{2} \log\left(\frac{3}{2}\right)$;

(c) $\log \tan\left(\frac{\pi}{4}+\frac{b}{2a}\right)$;

(d) $\frac{8p}{27}\left\{\left(1+\frac{9a}{4p}\right)^{\frac{3}{2}}-1\right\}$;

(e) $6a$;

(f) $4(a+b)-\frac{4ab}{(a+b)}$;

(g) $a\frac{(a+m^2)^{\frac{1}{2}}}{m}$;

(h) $\frac{a}{2}\{2\pi(1+4\pi^2)^{\frac{1}{2}}+\log(2\pi+(1+4\pi^2)^{\frac{1}{2}})\}$.

2.2.2 Determine the length of the cycloid

$$x=a(t-\sin t), \quad y=a(1-\cos t)$$

between the points 0 and t, $t<2\pi$.

2.2.3 Determine the length of the curve $y=\log x$ from its intersection with the x-axis to the point with abscissa x $(x>1)$.

ANSWER:

$$s = (x^2+1)^{\frac{1}{2}} - \sqrt{2} - \log\left\{\frac{(x^2+1)^{\frac{1}{2}}+1}{x(\sqrt{2}+1)}\right\}.$$

2.2.4 The parabolae $y^2 = ax$ $(a>0)$, $x^2 = by$ $(b>0)$ define a region G in the x,y-plane. Determine the circumference of G when

(a) $a=b$; (b) $a \neq b$.

2.2.5 Sketch the curve $x=\cos^4 t$, $y=\sin^4 t$ and determine the length of its circumference.

2.2.6 Sketch the curve

$$x = \cos t \sin^2 t\,, \quad y = \sin t\,(1+\cos^2 t)$$

and determine its length between the two points $t=0$ and $t=\frac{1}{2}\pi$.

2.2.7 Let $s(a)$ be the length of the curve $y=e^{ax}$, $a>0$, between the points $x=-1$ and $x=1$.

(a) Find an analytic expression for $s(a)$.

(b) Calculate the limit $\lim_{a\to 0^+} s(a)$.

ANSWER:

(a) $s(a) = \frac{1}{a^2}\{(1+a^2e^{2a})^{\frac{1}{2}} - (1+a^2e^{-2a})^2 + \log\left\{\frac{1+(1+a^2e^{-2a})^{\frac{1}{2}}}{1+(1+a^2e^{2a})^{\frac{1}{2}}}\right\} + 2$

(b) $\lim_{a\to 0^+} s(a) = 2.$

2.2.8 For the curve

$$y = ae^x + \frac{1}{4a}e^{-x} - b\,, \quad a>0, \quad b>1\,,$$

prove that

(a) the curve cuts the x-axis in two points $P_1(x_1, 0)$ and $P_2(x_2, 0)$;

(b) the length of the curve between the points P_1 and P_2 equals $2(b^2-1)^{\frac{1}{2}}$.

HINTS:

(a) The points of intersection of the curve with the x-axis are determined by the equation

$$ae^{2x}-be^{x}+\frac{1}{4a}=0. \qquad (*)$$

(b) Use the values of e^x given by (*).

2.2.9 Determine the circumference of the closed figure the sides of which are the curves

$$y=\frac{x^2}{4}-\log x;\ y=0;\ x=1\ ;\ x=2\,.$$

2.2.10 Find the length of the loop of the curve

$$9ay^2=x(3a-x)^2\,.$$

2.2.11 Sketch and find the lengths of the curves

(a) the epicycloid with $R=r$ (cardioid);

(b) the epicycloid with $R=2r$;

(c) the hypocycloid with $R=3r$.

2.3 TANGENT AND NORMAL

2.3.1 Find the equation of the tangent to the given curve at the point $t=t_0$ for

(a) $x=\dfrac{t^3-1}{3t^2-1},\quad y=\dfrac{t+1}{3t^2-1},\quad t_0=1\ ;$

(b) $x=\dfrac{t\sin t+2\cos t}{2+at-a^2},\quad y=\dfrac{t\cos t-2\sin t}{2+at-a^2},\quad t_0=a\,.$

2.3.2 Find the slope of the tangent to

$$x=\sin t,\qquad y=\frac{\sin t}{t}\,.$$

Use this information to sketch the curve defined by this parametric form.

ANSWER:

The curve consists of two branches which are mirror images of each other with respect to the y-axis.

The form of the branch, defined for positive t, is as follows:

As $t \to 0^+$, the point (x, y) approaches the point $(0, 1)$. As t moves from 0 to π, the point (x, y) describes a curve L_1 which lies in the first quadrant and ends at the origin.

During the variation of the parameter t in the interval $(\pi, 2\pi)$ the point (x, y) describes a loop L_2 in the third quadrant. For t in the interval $(2\pi, 3\pi)$, the point (x, y) describes the loop L_3 in the first quadrant. This loop is closer to the x-axis than L_1, etc. The branch corresponding to positive t consists of an infinity of such loops which are found either in the first or third quadrants.

2.3.3 Sketch the curve

$$x = te^{-t}, \quad y = (t+1)e^{-t}, \quad -\infty < t < +\infty,$$

and determine the equation of its tangent passing through the origin.

2.3.4 Consider the tangent to the curve $x^m y^n = 1$. Show that the section of this tangent lying between the two coordinate axes is divided in a fixed ratio by the point of contact between the curve and its tangent.

HINT:

The use of a parametric form for the curve shows that the ratio is n/m.

2.3.5 Determine the length of the tangent to

$$x = 2a \sin t + a \sin t \cos^2 t, \quad y = -a \cos^3 t$$

lying between the coordinate axes.

ANSWER:

$$2a(1+\sin^2 t)(\sin 2t)^{-1}.$$

2.3.6 The tangent at any point P to the curve $y=x^3$ cuts the x-axis at the point T, and the orthogonal projection of P onto the x-axis is N. Prove that $OT=2TN$, where O is the origin. Determine the corresponding relation for the curve $y=x^s$ and deduce from it the construction for the tangent when s is an integer.

2.3.7 Determine the equations of the tangents to $y=x^4-x+3$ which pass through the origin. Illustrate the result graphically.

2.3.8 Prove that the curve

$$y=k(x-a)(x-b)(x-c), \quad k, a, b, c \text{ real constants},$$

has the property that its tangent at the point with abscissa $\frac{1}{2}(a+b)$ passes through the point $(c, 0)$.

HINTS:

Show that the equation of the required tangent is

$$y+\tfrac{1}{8}k(a-b)^2(a+b-2c)=-\frac{k}{4}(a-b)^2\{x-\tfrac{1}{2}(a+b)\}.$$

This property of the cubic parabola has been described by T. Nakazawa (see: Mathematical Gazette, Note 39, Vol. 43, No. 345, 1959, pp. 203–4).

Can this property be generalized for parabolae of order greater than 3?

2.3.9 Consider the curve C given by

$$y=x^2-x^3.$$

(a) At the point P on C with abscissa b draw the tangent. Prove that this tangent cuts C at a point Q and determine its coordinates.

(b) At three arbitrary points P_1, P_2, P_3 on C draw the tangents to cut C at the points Q_1, Q_2, Q_3. Prove that Q_1, Q_2, Q_3 lie on a straight line if and only if P_1, P_2, P_3 lie on a straight line.

(c) Let P move along the curve C. Find the equation of the locus of the midpoint of the segment PQ.

HINT:

(a) Show that the coordinates of Q are $(1-2b, 2b(1-2b)^2)$.

(b) The condition for collinearity of P_1, P_2, P_3 is

$$\begin{vmatrix} b_1 & b_1^2 & -b_1^3 & 1 \\ b_2 & b_2^2 & -b_2^3 & 1 \\ b_3 & b_3^2 & -b_3^3 & 1 \end{vmatrix} = 0 .$$

(c) Show that the required equation is

$$y=1-9x+28x^2-28x^3 .$$

2.3.10 The tangent at the point P to the curve $y=f(x)$ cuts the x-axis at T and the y-axis at T_1, and the normal at P cuts the x-axis at N and the y-axis at N_1. Prove that

$$\frac{T_1N}{TN_1} = \left|\frac{dy}{dx}\right| .$$

2.3.11 Show that the curve C given by

$$x = a(2\cos t+\cos 2t), \quad y = a(2\sin t-\sin 2t)$$

has the following properties:

(a) The equations of the tangent and the normal at the point P, for which $t=t_0$, are

$$x\sin \tfrac{1}{2}t_0+y\cos \tfrac{1}{2}t_0 = a\sin \tfrac{3}{2}t_0 ,$$
$$x\cos \tfrac{1}{2}t_0-y\sin \tfrac{1}{2}t_0 = 3a\cos \tfrac{3}{2}t_0 .$$

(b) The tangent at P cuts C in the points Q and R with parametric values $-\frac{1}{2}t_0$ and $\pi-\frac{1}{2}t_0$.

(c) $QR=4a$.

(d) The tangents to C at Q and $t=\frac{1}{2}(\pi-t_0)$ are perpendicular and intersect at a point on the circle $x^2+y^2=a^2$.

2.3.12 Consider the function

$$f(x) = x^a(1-x)^{\frac{1}{2}} , \quad a>0 .$$

Show that for $a>1$ the tangent to the curve $y=f(x)$ at the point

$x=0$ is the line $y=0$, and that for $a<1$ the tangent at the same point is the line $x=0$.

HINT:
Use $f'(x)$.

2.3.13 Show that the length of the section of the tangent to the curve (tractrix)

$$x = \int_0^t \frac{dv}{\cos v} - \sin t, \quad y = \cos t,$$

between the point of contact and the point of intersection with the x-axis is constant.

2.3.14 Let T be a fixed point on the tangent to the curve C at M. As M moves along C, T describes a curve C'. Determine the equation of C' for

(a) $x=a\cos t, \quad y=a\sin t$;
(b) $x=a\cos t, \quad y=b\sin t$;
(c) $x=t, \quad y=t^n$;
(d) The curve of problem **2.3.13**.

2.3.15 Determine those tangents to the cycloid

$$x = a(t-\sin t), \quad y = a(1-\cos t), \quad a>0,$$

which are also normals to the cycloid.

ANSWER:
The equations of the tangent and the normal to the cycloid at the point $t=t_1$ and $t=t_0$ are

$$y-y_1 = \cot\frac{t_1}{2}(x-x_1),$$

$$y-y_0 = -\tan\frac{t_0}{2}(x-x_0).$$

These curves coincide when their slopes and intercepts on the y-axis coincide, i.e., when

$$\cot\frac{t_1}{2} = -\tan\frac{t_0}{2}, \tag{1}$$

and

$$(1-\cos t_1)-\cot\frac{t_1}{2}(t_1-\sin t_1)=(1-\cos t_0)+\tan\frac{t_0}{2}(t_0-\sin t_0) \tag{2}$$

are satisfied.

From (1), it follows that

$$t_0 = t_1-(2k+1)\pi, \quad k=0, \pm 1, \ldots ; \tag{3}$$

substitution in (2) and rearrangement give

$$2\cos t_1+(2k+1)\pi\cot\frac{t_1}{2} = 2\sin t_1\cot\frac{t_1}{2},$$

which reduces to

$$\tan\frac{t_1}{2} = (2k+1)\frac{\pi}{2}. \tag{4}$$

Equations (3) and (4) determine t_1 and t_0.

2.3.16 Show that no tangent to the parabola $y = x^2$ coincides with a normal to the same parabola.

2.3.17 When do the tangents to the parabola $y=x^2$ coincide with the normals to the parabola $y=-\frac{1}{2}x^2$.

2.3.18 A tangent to the ellipse $b^2x^2+a^2y^2=a^2b^2$ cuts the x-axis at P and the y-axis at Q. Find the maximum and the minimum values for the length of PQ.

2.3.19 Show that if $r=f(\theta)$ is the equation of a curve in polar coordinates, then

$$rr_0\cos(\theta-\theta_0) = r_0^2+rr_0'\sin(\theta-\theta_0), \quad r_0' = \mathrm{d}f(\theta_0)/\mathrm{d}\theta,$$

is the equation of the tangent to this curve at the point (r_0, θ_0).

2.3.20 Show that all normals to the curve

$$x = a(\cos t+t\sin t), \quad y = a(\sin t-t\cos t)$$

are equally distant from the origin.

2.3.21 Determine the length of the normal to

$$x = (a+b)\sin t - a\sin^3 t, \quad y = (a-b)\cos t + a\sin^2 t\cos t,$$

lying between the coordinate axes.

2.3.22 Determine the normals to the ellipse

$$x = a\cos t, \quad y = b\sin t$$

which are at a maximum distance from the centre of the ellipse and find this distance.

2.4 CURVATURE

2.4.1 Evaluate the radius of curvature ρ at the given point M:

(a) $x=t$, $y=t^3$, $M=(1, 1)$;

(b) $x=t^2$, $y=t^3$, $M=(1, 1)$;

(c) $y=a\cosh\dfrac{x}{a}$, $M=(0, a)$;

(d) $y=a\log\cos\dfrac{x}{a}$, M arbitrary ;

(e) $x=a\,e^{-t}(\cos t-\sin t)$, $y=a\,e^{-t}(\cos t+\sin t)$, M arbitrary ;

(f) $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$, M arbitrary .

ANSWER:

(a) $\frac{5}{3}\sqrt{10}$; (b) $\frac{13}{6}\sqrt{13}$; (c) a; (d) $a\left(\cos\dfrac{x}{a}\right)^{-1}$;

(e) $2a\,e^{-t}$; (f) at.

2.4.2 Determine the length and the radius of curvature of

(a) $x=t-\dfrac{t^3}{3}$, $y=t^2$;

(b) $x=t-e^{2t}$, $y=e^t(1-e^{2t})^{\frac{1}{2}}$.

2.4.3 Consider the curve C given by

$$y = \arcsin(\sqrt{x}) - \sqrt{1-x}\,.$$

Does the relation $\rho = \sin\alpha$, where ρ is the radius of curvature of C at

a point M and α is the angle between the tangent at M and the x-axis, hold for C?

2.4.4 Calculate the curvature for the curve defined by the differential equation

$$P(x, y)\mathrm{d}x+Q(x, y)\mathrm{d}y=0\,.$$

HINT:

Determine $y'=\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $y''=\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Then use the formula

$$\kappa=|y''|(1+(y')^2)^{-\frac{3}{2}}$$

to obtain

$$\kappa=\{P(QQ_x-PQ_y)+Q(PP_y-QP_x)\}(P^2+Q^2)^{-\frac{3}{2}}\,,$$

where $P_x=\dfrac{\partial P}{\partial x}$, etc.

2.4.5 Find the point at which the curve $y=2^x$ has maximum curvature.

ANSWER:

$$x=-\frac{\log(\sqrt{2}\log 2)}{\log 2}\,,\qquad y=(\sqrt{2}\log 2)^{-1}\,.$$

2.4.6 Find the point at which the curve $y=e^x$ has maximum curvature.

2.4.7 Determine the relationships between the constants a, b, c, p, q and r such that the curves

(a) $y_1=p+qx+rx^2$ and (b) $y_2=a+be^{cx}$

coincide with equal curvature at the point $x=0$. If

$$y_1=1+x-\tfrac{1}{2}x^2$$

is a curve of type (a), what is the corresponding curve of type (b)?

ANSWER:

$$y=2-e^x .$$

2.4.8 Determine the relation between the constants a, b, c and d such that the curves

$$y=e^x \quad \text{and} \quad y=(ax+b)(cx+d)^{-1}$$

coincide with equal curvature at the point $x=0$.

2.4.9 Determine the radius of curvature ρ of the curves $y=a\cosh(x/a)$ at an arbitrary point P. Let the normal of this curve at P cut the x-axis at Q. Prove that $PQ=\rho$.

2.4.10 Show that if a curve is defined implicitly by the equation

$$\phi(x, y)=0 ,$$

then the formula for its curvature is given by

$$\kappa=-(\phi_{xx}\phi_y^2-2\phi_{xy}\phi_x\phi_y+\phi_{yy}\phi_x^2)(\phi_x^2+\phi_y^2)^{-\frac{3}{2}} ,$$

where $\phi_x=\dfrac{\partial\phi}{\partial x}$, $\phi_y=\dfrac{\partial\phi}{\partial y}$, etc.

2.4.11 Among the family of curves $(2x+y-1)^2-2pxy=0$, for which p is the family parameter, determine the curves for which the curvature at $(0, 1)$ is $\frac{1}{4}$.

2.4.12 From a point P draw the tangents PA and PB to the parabola $y=x^2$. Show that

$$\frac{PA}{PB}=\left(\frac{\rho_A}{\rho_B}\right)^{\frac{1}{2}} ,$$

where ρ_A and ρ_B are the radii of curvature of the parabola at the points A and B, respectively.

2.4.13 Sketch the curve $y=x^x$. Determine the radius of curvature $\rho(x)$. Use this to determine the radius of curvature at the minimum on the given curve, and evaluate $\lim_{x\to 0}\rho(x)$.

2.4.14 Determine numerically the point of intersection of the curve C, given by $y=-x-\log x$, with the x-axis. Sketch C. At what point does C attain its maximum curvature? Determine this curvature.

2.4.15 Find expressions, in polar coordinates r and θ, for the element of arc length $\mathrm{d}s$ and the curvature κ of a curve.

ANSWER:

$$\mathrm{d}s=(r'^2+r^2)^{\frac{1}{2}}\mathrm{d}\theta\,, \qquad \kappa=\frac{r^2+2r'^2-rr''}{(r'^2+r^2)^{\frac{3}{2}}}\,,$$

where

$$r'=\frac{\mathrm{d}r}{\mathrm{d}\theta}\,, \quad r''=\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2}\,.$$

2.4.16 Determine the radius of curvature for the curves

(a) $r=a\theta$ (the Archimedian spiral) ;

(b) $r=a(1+\cos\theta)$;

(c) $r^2=a^2\cos 2\theta$, at $\theta=0$;

(d) $r\theta=a$;

(e) $r=a\,e^{m\theta}$;

where a is a constant.

2.4.17 Calculate the radius of curvature for the curve $y=y(x)$ defined by $s=f(x)$, where s is the arc length.

HINT:

Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{f'^2-1}\,.$

2.5 THE INTRINSIC EQUATION

2.5.1 Determine the intrinsic equation of the curve C defined by

$$s=a^{1-n}x^n\,.$$

HINT:

Use **2.4.17** to find ρ. Eliminate x from the equations for s and ρ to obtain

$$\rho = \frac{n}{n-1} s\left\{ n^2 \left(\frac{s}{a}\right)^{2(n-1)/n} - 1 \right\}^{\frac{1}{2}} .$$

2.5.2 Find the intrinsic equation of the curve

$$s = a \log \frac{x}{a} .$$

ANSWER:

$$\rho = a\{e^{-2s/a} - 1\}^{\frac{1}{2}} .$$

2.5.3 Determine each of the curves the arc lengths s of which are defined by the relationships

(a) $x = e^s$; (b) $s^2 = x$; (c) $\sin s = x$.

2.5.4 Determine the curve for which $\rho = f(\phi)$, where ϕ is the angle between the tangent and a fixed direction.

SOLUTION:

If x is the arc length of the curve, then $\rho = \mathrm{d}s/\mathrm{d}\phi$. Choosing the coordinate system so that the x-axis lies in the fixed direction, the tangent $\boldsymbol{t} = \{\cos \phi, \sin \phi\}$. Hence, the parametric equations for the curve are

$$x = \int_a^t f(\phi) \cos \phi \, \mathrm{d}\phi, \qquad y = \int_b^t f(\phi) \sin \phi \, \mathrm{d}\phi ,$$

where every choice of a and b defines a curve. All these curves are congruent.

2.5.5 Find the curves for which the following relationships between the radius of curvature ρ and the angle ϕ between the tangent to the curve and the x-axis hold:

(a) $\rho = c$; (b) $\rho = c(\sin \phi)^{-3}$; (c) $\rho = c\phi$; (d) $\rho = c \sin \phi$;

(e) $\rho = c(\cos \phi)^{-2}$;

where $c = \text{const}$.

HINT:

Use the result of **2.5.4**.

ANSWER:

(a) The circles $(x-x_0)^2+(y-y_0)^2=c^2$.

(b) The parabolae $(y-y_0)^2=2c\left(x_0-\frac{c}{2}-x\right)$.

(c) The involutes of the circles $(x-x_0)^2+(y-y_0)^2=c^2$.

(d) The cycloids $x=x_0-\frac{c}{4}\cos 2t$, $y=y_0+\frac{c}{2}t-\frac{c}{4}\sin 2t$.

(e) The catenaries

$$x=x_0+a\log\tan\left(\frac{\pi}{4}+\frac{t}{2}\right),\quad y=y_0+a(\cos t)^{-1}.$$

2.5.6 Repeat **2.5.5** for

$$\rho=ce^{m\phi},$$

where c and m are constants.

HINT:

Show that the curve is a logarithmic spiral.

2.5.7 Find the curves with the intrinsic equations

(a) $\rho=a+\frac{s^2}{a}$;

(b) $\rho^2=2as$;

(c) $\rho^2+s^2=16a^2$;

(d) $\rho=as+b$;

where a and b are constants.

ANSWER:

The general parametric equations for these curves are

$$x=\int_{a_0}^{t}\cos\phi\,ds,\quad y=\int_{b_0}^{t}\sin\phi\,ds,\quad \text{where } \phi=\int\frac{ds}{\rho(s)}.$$

(a) Here

$$\phi=a\int\frac{ds}{a^2+s^2}=\arctan\left(\frac{s}{a}\right).$$

Thus,

$$x = \int_{a_0}^{t} \frac{a}{\sqrt{a^2+s^2}}\,\mathrm{d}s = a \operatorname{arcsinh}\left(\frac{t}{a}\right) - a \operatorname{arcsinh}\left(\frac{a_0}{a}\right),$$

and

$$y = \int_{b_0}^{t} \frac{s}{\sqrt{a^2+s^2}}\,\mathrm{d}s = (t^2+a^2)^{\frac{1}{2}} - (b_0^2+a^2)^{\frac{1}{2}},$$

which are the parametric equations of a catenary with vertex at $(a \operatorname{arcsinh}(a_0/a), (b_0^2+a^2)^{\frac{1}{2}}+a)$.

For the remaining intrinsic equations show that they represent:
(b) involutes of circles of radius a, (c) cycloids, and (d) logarithmic spirals.

2.5.8 Find the intrinsic equation of the hypocycloid and the epicycloid.

ANSWER:

The equation $\dfrac{s^2}{a^2} + \dfrac{\rho^2}{b^2} = 1$ is the intrinsic equation of the epicycloid, hypocycloid or cycloid depending on whether $a^2 > b^2$, $a^2 < b^2$ or $a^2 = b^2$.

2.5.9 Find the intrinsic equation of an equiangular spiral.

2.5.10 On the normal to the curve C mark off a constant distance a. The locus of this point describes a curve C^* which is said to be *parallel* to C as it never intersects C. Determine the radius of curvature ρ^* of C^*.

HINT:

The radii of curvature ρ and ρ^* of C and C^* at corresponding points on the normal to C satisfy

$$\rho^* = \rho \pm a\,.$$

2.5.11 Let $s = f(\rho)$ be the intrinsic equation of a curve C. Determine the intrinsic equation to a curve parallel to C.

HINT

Use the fact that $\frac{ds^*}{\rho^*} = \frac{ds}{\rho}$ to obtain

$$s^* = \int \frac{\rho^* f'(\rho^* \pm a)}{\rho^* \pm a} \, d\rho^* .$$

2.6 CIRCLES OF CURVATURE AND EVOLUTES

2.6.1 Determine the centre of curvature for the curves (where k is a constant):

(a) $xy = k^2$ at the point (k, k) ;

(b) $y = x \log x$ at the point where $dy/dx = 0$.

ANSWER

(a) $(2k, 2k)$;

(b) $\left(\frac{1}{e}, 0\right)$

2.6.2 Let (r, s) be a point on the astroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

and (p, q) the position of the centre of curvature of the astroid at (r, s). Prove that

$$(r^{\frac{1}{3}} + s^{\frac{1}{3}})^3 = p + q .$$

2.6.3 Let P be the centre of curvature of $y = x^2$ at the point M with abscissa x. Find the locus of the midpoint of PM.

2.6.4 If C is given in polar form, $r = r(\theta)$, show that the centre of curvature of C has (Cartesian) coordinates

$$X = r \cos \theta - \frac{r(\dot{r}^2 + r^2) \sin \theta}{r^2 + 2\dot{r}^2 - r\ddot{r}} ,$$

$$Y = r \sin \theta - \frac{r(\dot{r}^2 + r^2) \cos \theta}{r^2 + 2\dot{r}^2 - r\ddot{r}} ,$$

where the dot denotes differentiation with respect to θ.

2.6.5 Find the radius of curvature and the centre of curvature for (where k is a constant)

(a) $r=k(1+\cos\theta)$;

(b) $r\theta=4$ (hyperbolic spiral) at $\frac{3}{4}\pi$;

(c) $r=k\theta$ (the Archimedean spiral).

2.6.6 Find the equation of the circle of curvature for

(a) the parabola $y=x^2$ at its vertex ;

(b) the cardioid $x=(1-t^2)/(1+t^2)^2$, $y=2t/(1+t^2)^2$.

2.6.7 Determine the equation of the circle of curvature and the evolute of

$$x = \frac{t^2}{t^2+1}, \qquad y = \frac{t^3}{t^2+1}.$$

ANSWER:

The circle of curvature is

$$x^2+y^2+\frac{t^2(t^2+6)}{3}x-\frac{8t}{3}y-\frac{t^4}{3}=0$$

and the evolute

$$x+\tfrac{9}{16}y^2+\frac{27}{2^9}y^4=0.$$

2.6.8 Find the evolutes of the curves (where a and b are constant)

(a) $x=a\cosh t$, $y=b\sinh t$;

(b) $y^2(2a-x)=x^3$;

(c) $y=a\cosh\left(\frac{x}{a}\right)$;

(d) $x=a\left\{\log\left(\tan\frac{t}{2}\right)+\cos t\right\}$, $y=a\sin t$;

(e) $r=e^{a\theta}$;

(f) $y^2=2px$;

(g) $x=a\cos t$, $y=b\sin t$;

(h) $y=x^n$, $n\geqslant 2$;

(i) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$.

ANSWER:

(a) $(ax)^{\frac{2}{3}} - (bx)^{\frac{2}{3}} = (a^2+b^2)^{\frac{1}{3}}$;

(b) $\left(\frac{3y}{8}\right)^4 + 6a^2\left(\frac{3y}{8}\right)^2 + 3a^2x = 0$;

(c) $x = a\left(\log t - \frac{t}{2}\right), \quad t = \frac{y+(y^2-4a^2)^{\frac{1}{2}}}{2a}$;

(d) $y = a\cosh\left(\frac{x}{a}\right)$;

(e) a logarithmic spiral;

(f) $27py^2 = 8(x-p)^3$;

(g) $x = \frac{a^2-b^2}{a}\cos^3 t, \quad y = \frac{-a^2+b^2}{b}\sin^3 t$;

(h) $x = \frac{n-2}{n+1}t - \frac{n^2}{n-1}t^{2n-1}, \quad y = \frac{2n-1}{n-1}t^n + \frac{1}{n(n-1)}t^{2-n}$;

(i) $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 1$.

2.6.9 Determine the length of the evolute of the hyperbola

$$x = a\cosh t, \qquad y = b\sinh t$$

between the points $P\left(\frac{c^2}{a}, 0\right)$ and $Q\left(2\sqrt{2}\,\frac{c^2}{a}, \frac{c^2}{b}\right)$, where $c^2 = a^2+b^2$.

2.6.10 Find the radius of curvature of the evolute of a curve C, $\boldsymbol{r} = \boldsymbol{r}(s)$.

HINT:

Differentiate the equation of the evolute

$$\boldsymbol{r}^* = \boldsymbol{r}^*(s^*) = \boldsymbol{r} + \rho\boldsymbol{n}$$

with respect to s to show that

$$\boldsymbol{t}^* = \boldsymbol{n}, \qquad \frac{\mathrm{d}s^*}{\mathrm{d}s} = \frac{\mathrm{d}\rho}{\mathrm{d}s}. \tag{*}$$

Use (*) to show that

$$\rho^* = \rho\frac{\mathrm{d}\rho}{\mathrm{d}s}, \qquad \boldsymbol{n}^* = -\boldsymbol{t}.$$

2.6.11 Consider the curve $y=f(x)=2\sinh(x/2)$.

(a) Find the point (x_0, y_0) in the half-plane $x>0$ at which the curve $y=f(x)$ attains its greatest curvature. What is the corresponding radius of curvature?

(b) Draw the tangent from the point (2, 0) to the curve. Determine to two decimal places the coordinates of the point of contact (x_1, y_1). What is the radius of curvature of $y=f(x)$ at this point?

(c) Find the parametric form for the evolute of $y=f(x)$. Without integrating but using the results of (a) and (b), determine the length of the evolute between the centres of curvature of $y=f(x)$ at (x_0, y_0) and (x_1, y_1).

ANSWER:

(a) At the point $(2t, 2\sinh t)$,

$$\kappa=\tfrac{1}{2}\sinh t(1+\cosh^2 t)^{\frac{3}{2}}.$$

Hence $x_0\doteqdot 1.752$, $y_0=2$ and $\rho_0=6\sqrt{3}$.

(b) Since the tangent at (x_1, y_1) passes through (2, 0), its follows that $\sinh t_1=(t_1-1)\cosh t_1$.

Using Newton's method with $t_0=0$ as first approximation, one obtains that $x_1\doteqdot 3.92$, $y_1\doteqdot 6.97$ and $\rho_1\doteqdot 30.5$.

(c) The evolute is

$$x=2t-2(1+\cosh t)\operatorname{\dot{c}oth} t,$$
$$y=4\cosh^2 t(\sinh t)^{-1}.$$

Using second equation of (*) in **2.6.10**, the required length is

$$|s_1^*-s_0^*|=|\rho_1-\rho_2|\doteqdot 20.11.$$

2.6.12 The evolute of the evolute of a given curve C is called the *second evolute* of C. Determine the radius of curvature ρ^{**} of the second evolute of C in terms of ρ.

HINT:

Use **2.6.10** to show that

$$\rho^{**}=\rho\{(\rho')^2+\rho\rho''\},$$

where the dash denotes differentiation with respect to s.

2.6.13 The evolute of the second evolute of C is called the *third evolute* of C. Prove that the radius of curvature ρ^{***} of the third evolute is

$$\rho^{***} = \rho(\rho')^3 + 4\rho^2\rho'\rho'' + \rho^3\rho''' ,$$

where the dash denotes differentiation with respect to s.

2.6.14 On the tangent at an arbitrary point M of a curve C, consider the point P which lies at a fixed distance k from M. The locus of P describes a curve C^*. Prove that the line, connecting the point P and the centre of curvature of C at M, is normal to C^*.

HINT:
Use the fact that the dot product of two orthogonal vectors is zero.

2.7 INVOLUTES

2.7.1 Prove that the involutes of the curve $r = r(t)$ have the parametric form

$$X = x(t) + K\dot{x}(t), \qquad Y = y(t) + K\dot{y}(t) ,$$

where

$$K = \frac{\int_t^{t_0} \{(\dot{x})^2 + (\dot{y})^2\}^{\frac{1}{2}} \, dt}{(\dot{x})^2 + (\dot{y})^2} .$$

2.7.2 Find the involute of the circle $x^2 + y^2 = a^2$ which passes through the point $(a, 0)$.

ANSWER:

$$x = a \cos t + at \sin t ,$$

$$y = a \sin t - at \cos t .$$

2.7.3 Prove that the involutes of a cycloid are cycloids.

2.7.4 Consider the involutes of a curve $r = r(s)$. Use the parametric form for the involute to prove that the normal to an involute coincides

with the tangent to $r=r(s)$. Find an expression for the radius of curvature of an involute of $r=r(s)$.

2.7.5 Consider the curve

$$x=t-\sinh t\cosh t,\quad y=2\cosh t\,.$$

(a) Find the curvature of C.
(b) Determine the evolute and involutes of C in parametric form.

ANSWERS:
(a) $\kappa=1/(2\sinh t\cosh^2 t)$;
(b) The evolute is

$$x=t-3\sinh t\cosh t,\qquad y=4\cosh t-2\cosh^2 t\,,$$

and the involutes are

$$x=t+(c-1)\tanh t,\qquad y=\cosh t+(1-c)/\cosh t\,,$$

where c is an arbitrary constant.

2.8 ENVELOPES

2.8.1 Find the envelopes of the following families of curves, where λ is the family parameter:
(a) $y=\lambda-\lambda^2x$;
(b) $y=\lambda^2-\lambda x$;
(c) $y=27\lambda-\lambda^3x^2$;
(d) $x^2-2\lambda x-y+\lambda+\lambda^2=0$;
(e) $y=\lambda x^2+1/(4\lambda)$;
(f) $x^2+xy+y^2-2\lambda x-2\lambda y+\lambda^2=0$;
(g) $(x-\lambda)^2-2\lambda(x+y)+\lambda^2=0$;
(h) $x-2\lambda y+\lambda^2=0$;
(i) $(x-\lambda)^2+y^2=-4(\lambda+1)$;
(j) $(1+\lambda)^2y-x(x+\lambda+\lambda^2)=0$.

ANSWERS:
(a) $x^2y=\frac{1}{4}$; (b) $y+\frac{1}{4}x^4=0$; (c) $xy=\pm54$; (d) $4x-4y-1=0$;
(e) $x\pm y=0$; (f) $x=0,\ y=0$; (g) $x=0,\ y=0$; (h) $x\pm y=0$;
(i) $y^2+4x=0$; (j) $4x-4y-1=0$.

2.8.2 The centre of a circle with constant radius k moves along the curve C given by

$$x = f(\lambda), \quad y = g(\lambda).$$

Determine the envelope of the corresponding family of circles.

HINT:

Show that when $\lambda = s$, the arc length of C, the envelope becomes

$$x = f(s) \pm kg'(s), \quad y = g(s) \pm kf'(s),$$

where the dash denotes differentiation with respect to s.

2.8.3 Find the envelopes of the following families of curves:

(a) The straight lines, which form with the x- and y-axes the triangles of constant area $2k^2$;

(b) The straight lines on which the coordinate axes cut off intervals of constant length k.

ANSWER:

(a) $xy = a^2$, $x > 0, y > 0$;

(b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

2.8.4 Find the length and curvature of the envelope of the family of straight lines

$$x \cos \lambda + y \sin \lambda - p(\lambda) = 0,$$

where λ is the family parameter.

ANSWER:

$$s = \int_{t_0}^{t} (p + p'') \mathrm{d}\lambda, \quad \kappa = (p + p'')^{-1},$$

where the dash denotes differentiation with respect to λ.

2.8.5 Prove that

$$y^2(x + y - 1) = x^3$$

is the envelope of the family of straight lines

$$(\lambda^3 - 3\lambda)x + (\lambda^3 + 2)y = \lambda^3 .$$

2.8.6 Show that

$$x+y\pm\sqrt{2}\,a=0$$

is the envelope of the family of circles

$$x=\lambda+a\cos t, \quad y=-\lambda+a\sin t\,,$$

where λ is the family parameter.

2.8.7 Find the envelope of a fixed radius of a circle of radius k rolling along the x-axis.

2.8.8 Find the envelope of a fixed radius of a circle of radius k which rolls along

(a) the outside of a circle of radius b $(b>k)$;

(b) the inside of a circle of radius b $(b>k)$.

2.8.9 Consider the family of curves

$$x=\lambda+t^2\,, \quad y=\lambda t\,, \qquad (*)$$

where λ is the family parameter. Find the envelope of the curves $(*)$:

(a) by elimination of the parameter t, and

(b) without elimination of the parameter t.

Interchange the role of λ and t in $(*)$, and redetermine the envelope using (a) and (b).

HINT:

In each of the four cases, the envelope is

$$4x^3-27y^2=0\,.$$

2.8.10 Find the envelope of the family of curves

(a) $x=u^2+v^2\,, \quad y=uv(u+v)$;

(b) $x=u\ +v^2\,, \quad y=u^2+v$;

when (i) u is the family parameter, and (ii) v is the family parameter.

2.8.11 Under what conditions does the envelope of the family of curves

$$x = x(u, v), \quad y = y(u, v), \tag{*}$$

with u as family parameter, coincide with the envelope of (*) with v as family parameter.

ANSWER:
Always, if the functions $x(u, v)$ and $y(u, v)$ are such that
(a) $\boldsymbol{r}_u = \partial \boldsymbol{r}/\partial u \neq 0$, $\boldsymbol{r}_v = \partial \boldsymbol{r}/\partial v \neq 0$, and
(b) the curve $|\boldsymbol{r}_u \times \boldsymbol{r}_v| = 0$ is neither a curve $u = \text{const.}$ nor a curve $v = \text{const.}$

2.8.12 Find the condition which ensures that the envelope of the family

$$A(u, v)x + B(u, v)y + C(u, v) = 0, \tag{*}$$

with u as family parameter, coincides with the envelope of (*) with v as family parameter.

Part II Three-Dimensional Differential Geometry

CHAPTER 3

ELEMENTS OF THE THEORY OF SPACE CURVES

3.1 THE CONCEPT OF CURVE

The point set $\boldsymbol{r} = \{x(t), y(t), z(t)\}$, $a \leqslant t \leqslant b$, defines a *smooth curve* when the functions $x(t)$, $y(t)$ and $z(t)$ are continuous on their interval of definition $a \leqslant t \leqslant b$, and when the first derivatives of $x(t)$, $y(t)$ and $z(t)$ exist, are continuous, and are non-zero on $a \leqslant t \leqslant b$.

The equations

$$x = x(t), \quad y = y(t), \quad z = z(t), \tag{3.1.1}$$

where x, y and z define the coordinates of the position vector which determines the smooth curve, are called *the parametric representation* or *parametric form* of this curve, while

$$\boldsymbol{r} = \boldsymbol{r}(t) = \{x(t), y(t), z(t)\}, \qquad a \leqslant t \leqslant b,$$

is called the *vector form* of this curve.

3.2 THE UNIT TANGENT

The vector $\dot{\boldsymbol{r}}(t)$ has the direction of the tangent to the curve $\boldsymbol{r} = \boldsymbol{r}(t)$ at t. If $\boldsymbol{r} = \boldsymbol{r}(t)$ is a smooth curve, then the unit tangent $\boldsymbol{t}$ is given by

$$\boldsymbol{t} = \dot{\boldsymbol{r}}/|\dot{\boldsymbol{r}}| = \dot{\boldsymbol{r}}/(\dot{\boldsymbol{r}}^2)^{\frac{1}{2}}, \qquad \dot{\boldsymbol{r}}^2 = \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}. \tag{3.2.1}$$

3.3 THE ARC LENGTH

The arc length s of a smooth curve $\boldsymbol{r} = \boldsymbol{r}(t)$ between the points $t = a$ and $t = b$ is defined by

$$s = \int_a^b (\dot{\boldsymbol{r}}^2)^{\frac{1}{2}} \mathrm{d}t\,, \qquad \dot{\boldsymbol{r}}^2 = \dot{\boldsymbol{r}} \,.\, \dot{\boldsymbol{r}}\,. \tag{3.3.1}$$

The parameter t of $\boldsymbol{r} = \boldsymbol{r}(t)$ can be replaced by some other parameter. Let $t = f(u)$, then the new vector form for the curve is $\boldsymbol{r} = \boldsymbol{r}(f(u)) = \boldsymbol{r}^*(u)$. One such value for t is the arc length s, measured from some fixed point. Then, and only then, $(\boldsymbol{r}')^2 = 1$ and $\boldsymbol{t} = \boldsymbol{r}'$, where the dash denotes differentiation with respect to s.

3.4 THE OSCULATING PLANE

On $\boldsymbol{r} = \boldsymbol{r}(t)$, consider the three points t, $t+h$ and $t+k$. In general, these three points determine a plane.

As $h \to 0$ and $k \to 0$, the limiting position of the plane always exists, if $\ddot{\boldsymbol{r}}(t) \neq 0$; it is called the *osculating plane* of $\boldsymbol{r} = \boldsymbol{r}(t)$ at t. It is determined by the point t and the vectors $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$. The vector equation of the osculating plane is

$$[\boldsymbol{R} - \boldsymbol{r}(t), \dot{\boldsymbol{r}}(t), \ddot{\boldsymbol{r}}(t)] = 0\,, \tag{3.4.1}$$

where $\boldsymbol{R}$ is the position vector of a point on the plane.

When $\boldsymbol{r} = \boldsymbol{r}(t)$ is defined by the parametric form

$$x^i = x^i(t) \qquad (i = 1, 2, 3\,),$$

the equation of the osculating plane becomes

$$\begin{vmatrix} X^1 - x^1(t) & X^2 - x^2(t) & X^3 - x^3(t) \\ \dot{x}^1(t) & \dot{x}^2(t) & \dot{x}^3(t) \\ \ddot{x}^1(t) & \ddot{x}^2(t) & \ddot{x}^3(t) \end{vmatrix} = 0\,, \tag{3.4.2}$$

where $\boldsymbol{R} = \{X^1, X^2, X^3\}$.

At points where $\ddot{\boldsymbol{r}}(t) = 0$, the osculating plane is not defined. Such points t are called *turning points.*

3.5 THE TANGENT, NORMAL AND BINORMAL

If the arc length s is chosen as parameter, the vectors $\boldsymbol{r}'$ and $\boldsymbol{r}''$ lie in the osculating plane. Moreover, the vector $\boldsymbol{r}''$ is perpendicular to $\boldsymbol{r}'$ and normal to $\boldsymbol{r} = \boldsymbol{r}(s)$. The unit vector

$$\boldsymbol{n} = \frac{\boldsymbol{r}''}{(\boldsymbol{r}''.\ \boldsymbol{r}'')^{\frac{1}{2}}} \tag{3.5.1}$$

is called the *unit (principal) normal*. The vector $\boldsymbol{b} = \boldsymbol{t} + \boldsymbol{n}$, which is also normal to the curve, is called the *unit binormal*. The unit vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ form a right-handed system and determine three planes at every point t of the curve. The *osculating plane*, determined by $\boldsymbol{t}$ and $\boldsymbol{n}$, was mentioned in the previous section. The unit vectors $\boldsymbol{n}$ and $\boldsymbol{b}$ define a plane which is normal to $\boldsymbol{t}$ and for this reason is called the *normal plane* of the curve at t. The plane which is defined by the vectors $\boldsymbol{t}$ and $\boldsymbol{b}$ is called the *rectifiable plane* of the curve at t.

The following relationships hold, since the vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ form an orthonormal set:

$$\begin{gathered} \boldsymbol{t}^2 = \boldsymbol{b}^2 = \boldsymbol{n}^2 = 1\,, \\ \boldsymbol{t}.\boldsymbol{n} = \boldsymbol{n}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{t} = 0\,, \\ \boldsymbol{t}\times\boldsymbol{n} = \boldsymbol{b},\ \boldsymbol{n}\times\boldsymbol{b} = \boldsymbol{t},\ \boldsymbol{b}\times\boldsymbol{t} = \boldsymbol{n}\,. \end{gathered} \tag{3.5.2}$$

If the parameter t is not the arc length s, then the vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are determined by construction of three vectors parallel to $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ and division of these vectors by their respective lengths. For a given curve $\boldsymbol{r} = \boldsymbol{r}(t)$, three vectors parallel to $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are $\dot{\boldsymbol{r}}$, $\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}\times\dot{\boldsymbol{r}}$, $\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}$.

The straight lines which pass through the point t of $\boldsymbol{r} = \boldsymbol{r}(t)$ in the directions of $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are called the *tangent*, *normal* and *binormal* to the curve at t; their corresponding equations are

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{r} + \lambda\dot{\boldsymbol{r}} \\ \boldsymbol{R} &= \boldsymbol{r} + \lambda(\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}\times\dot{\boldsymbol{r}})\,, \\ \boldsymbol{R} &= \boldsymbol{r} + \lambda(\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}})\,, \end{aligned}$$

where λ is a parameter defining points on the respective lines.

3.6 THE SERRET–FRENET FORMULAE

The Serret–Frenet formulae are basic to the study of the differential geometry of space curves. These formulae are

$$\boldsymbol{t}' = \kappa\boldsymbol{n}\,, \quad \boldsymbol{n}' = \tau\boldsymbol{b} - \kappa\boldsymbol{t}\,, \quad \boldsymbol{b}' = -\tau\boldsymbol{n}\,, \tag{3.6.1}$$

where κ and τ, called, respectively, the curvature and torsion of the curve $\boldsymbol{r} = \boldsymbol{r}(s)$, are defined by

$$\kappa^2 = (\boldsymbol{r}'')^2 , \qquad \tau = \frac{[\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}''']}{(\boldsymbol{r}'')^2} . \tag{3.6.2}$$

If the parameter s is replaced by t, the equations (3.6.2) become

$$\kappa^2 = \frac{(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}})^2}{(\dot{\boldsymbol{r}}^2)^3} , \qquad \tau = \frac{[\dot{\boldsymbol{r}}, \ddot{\boldsymbol{r}}, \dddot{\boldsymbol{r}}]}{(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}})^2} . \tag{3.6.3}$$

The *radius of curvature* ρ is defined by $1/\kappa$ and the *radius of torsion* σ by $1/\tau$.

An important result, sometimes called *the Fundamental Theorem of space curves*, is: *If two single-valued functions* $\kappa(s)$ *and* $\tau(s)$, $s > 0$, *are given, then there exists one and only one space curve, determined apart from its position in space, for which s is the arc length,* κ *the curvature, and* τ *the torsion.*

For this reason, $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are called the *intrinsic* (*natural*) *equations* of the curve $\boldsymbol{r} = \boldsymbol{r}(s)$.

3.7 THE SPHERICAL INDICATRIX

Consider unit vectors at the origin which are parallel to the vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ for the curve C, $\boldsymbol{r} = \boldsymbol{r}(s)$. The end points of these vectors describe curves on the unit sphere at the origin as s increases. The curve $\boldsymbol{r} = \boldsymbol{t}(s)$ is called *the spherical indicatrix of the tangents* to C, while $\boldsymbol{r} = \boldsymbol{n}(s)$ is called *the spherical indicatrix of the principal normals* to C and $\boldsymbol{r} = \boldsymbol{b}(s)$ *the spherical indicatrix of the binormals* to C. If the arc lengths of the curves $\boldsymbol{r} = \boldsymbol{t}(s)$ and $\boldsymbol{r} = \boldsymbol{b}(s)$ are denoted by s_t and s_b, respectively, then

$$\left|\frac{\mathrm{d}s_t}{\mathrm{d}s}\right| = \kappa , \qquad \left|\frac{\mathrm{d}s_b}{\mathrm{d}s}\right| = |\tau| .$$

Thus, one obtains a geometric interpretation for the curvature and torsion: The curvature is equal to the rate at which the tangent changes direction, while the torsion is equal to the rate at which the binormal changes direction.

3.8 THE CIRCLE OF CURVATURE AND THE OSCULATING SPHERE

The position vector $\boldsymbol{r}^* = \boldsymbol{r} + \rho \boldsymbol{n}$ determines *the centre of curvature.*

The circle with centre $\boldsymbol{r}^*$ and radius ρ which lies in the osculating plane is called *the circle of curvature*. This is the circle which is tangent to the curve C at the point in question, that is, the limiting circle, which passes through the points s, $s+h$ and $s+k$, as h and k tend to zero.

The sphere which is tangent to the curve $\boldsymbol{r}=\boldsymbol{r}(s)$ at the point s is called the *osculating sphere* at the point s. This is the limiting sphere, which passes through the four points s, $s+h$, $s+k$, $s+l$ as h, k and l tend to zero. The position vector $\boldsymbol{m}$ of the centre and the radius R of the osculating sphere are defined by

$$\boldsymbol{m} = \boldsymbol{r}+\rho\boldsymbol{n}+\rho'\sigma\boldsymbol{b}, \quad R^2 = \rho^2+(\rho'\sigma)^2 .$$

The vector $\boldsymbol{m}$ is called the *centre of spherical curvature* and R the *radius of spherical curvature.*

The line perpendicular to the osculating plane which passes through the centre of spherical curvature is called *the axis of spherical curvature*, or the *polar line* of the curve.

The curve $\boldsymbol{m}=\boldsymbol{m}(s)$ is called the *polar curve* of C (see [5]).

CHAPTER 4

PROBLEMS ON THREE-DIMENSIONAL SPACE CURVES

4.1 Find the lengths of the following space curves between the points $t=0$ and $t=t_0$:

(a) $\boldsymbol{r} = \left\{a\cos bt - b\cos at,\ a\sin bt + b\sin at,\ 4\dfrac{ab}{a+b}\cos\dfrac{a+b}{2}t\right\}$;

(b) $\boldsymbol{r} = \{\sin^2 t,\ \sin t\cos t,\ \log\cos t\}$.

ANSWER:

(a) $2abt_0$;

(b) $\log\tan\left(\dfrac{t_0}{2}+\dfrac{\pi}{4}\right)$.

4.2 Find the length of the curves:

(a) $\boldsymbol{r} = \left\{t+\dfrac{a^2}{t},\ t-\dfrac{a^2}{t},\ 2a\log\left(\dfrac{t}{a}\right)\right\}$, between the intersection of this curve with the x-axis and the point $t=t_0$;

(b) $y=\dfrac{x^2}{2a}$, $z=\dfrac{x^3}{6a^2}$, between the origin and the point P_0 with coordinate $\{x_0, y_0, z_0\}$.

ANSWER:

(a) $\sqrt{2}\left(t_0-\dfrac{a^2}{t_0}\right)$;

(b) x_0+z_0.

4.3 Show that any tangent to the curve

$$x^2 = 3y, \quad 2xy = 9z$$

makes a constant angle with a certain fixed direction. Determine the angle and the direction.

HINT:

Show that the unit tangent has the form

$$\boldsymbol{t} = \frac{1}{9+2t^2}\{9, 6t, 2t^2\}\,. \tag{1}$$

Using (1), form the vector equation

$$\boldsymbol{t}\cdot\boldsymbol{k} = c = \text{const.}\,,$$

where $\boldsymbol{k}=\{k_1, k_2, k_3\}$ is a fixed unit vector, to give:

$$2(c-k_3)t^2-6k_2t+9(c-k_1)=0\,.$$

Hence, deduce that the fixed direction is $\boldsymbol{k}=\frac{1}{2}\sqrt{2}\,\{1, 0, 1\}$ and that the angle between $\boldsymbol{k}$ and $\boldsymbol{t}$ is $\frac{1}{4}\pi$.

4.4 Find the points of the curve $\boldsymbol{r} = \left\{\frac{t^4}{4}, \frac{t^3}{3}, \frac{t^2}{2}\right\}$ at which the tangent is parallel to the plane $x+3y+2z=0$.

HINT:

Determine the values of t for which $\boldsymbol{t}\cdot\boldsymbol{N}=0$, where $\boldsymbol{N}$ is the normal to the plane; and hence show that the required points are $\{\frac{1}{4}, -\frac{1}{3}, \frac{1}{2}\}$ and $\{4, -\frac{8}{3}, 2\}$.

4.5 Show that the bisector of the angle between the tangent and the binormal to the curve $\boldsymbol{r}=\{3t, 3t^2, 2t^3\}$ is a fixed vector.

HINT:

Show that $\boldsymbol{t}+\boldsymbol{b} = \frac{1}{(1+2t^2)}\{1, 0, 1\}\,.$

4.6 Show that each of the following curves has the same radius of curvature and torsion:

(a) $\boldsymbol{r} = \{a(3t-t^2), 3at^2, a(3t+t^3)\}$;

(b) $\boldsymbol{r} = \left\{t, \frac{t^2}{2a}, \frac{t^3}{6a^2}\right\}$;

(c) $\boldsymbol{r} = \left\{t + \frac{a^2}{t}, t - \frac{a^2}{t}, 2a\log\frac{t}{a}\right\}$.

ANSWER:

(a) $3a(1+t^2)^2$;

(b) $(t^2+2a^2)^{\frac{3}{2}}/(4a^3)$;

(c) $2(a^2+t^2)/(at^2)$.

4.7 Find the curvature for the following curves:

(a) $\boldsymbol{r} = \{e^t \cos t,\ e^t \sin t,\ e^t\}$;

(b) $\boldsymbol{r} = \{4a \cos^3 t,\ 4a \sin^3 t,\ 3c \cos 2t\}$;

(c) $\boldsymbol{r} = \{a(t-\sin t),\ a(1-\cos t),\ 4a \sin (t/2)\}$.

ANSWER:

(a) $\frac{\sqrt{2}}{3} e^{-t}$;

(b) $a/(3[a^2+c^2] \sin 2t)$;

(c) $\left[1+\sin^2\left(\frac{t}{2}\right)\right]^{\frac{1}{2}}/(4a)$.

4.8 On the unit sphere with longitude φ and colatitude θ, find the radius of curvature and torsion for the curve $\varphi = \theta$.

HINT:

The equation of the curve is

$$\boldsymbol{r} = \{\cos t \sin t,\ \sin^2 t,\ \cos t\} .$$

This can be used to show that

$$\rho = \left\{\frac{(1+\sin^2 t)^3}{5+3 \sin t}\right\}^{\frac{1}{2}}, \qquad \sigma = \frac{5+3 \sin t}{6 \sin t} .$$

4.9 If the n-th derivative of the radius vector $\boldsymbol{r}(s)$ with respect to s is given by

$$\boldsymbol{r}^{(n)} = a_n \boldsymbol{t} + b_n \boldsymbol{n} + c_n \boldsymbol{b} ,$$

determine its $(n+1)$-th derivative.

HINT:

Use the Serret–Frenet formulae to show that

$$\boldsymbol{r}^{(n+1)} = a_{n+1}\boldsymbol{t} + b_{n+1}\boldsymbol{n} + c_{n+1}\boldsymbol{b} ,$$

where

$$\begin{aligned} a_{n+1} &= a_n' - \kappa b_n , \\ b_{n+1} &= b_n' + \kappa a_n - \tau c_n , \\ c_{n+1} &= c_n' + \tau b_n . \end{aligned}$$

4.10 Find the curve generated by the point of intersection between the tangent to a circular helix and a plane which is orthogonal to the axis of the given helix.

HINT:

Let the circular helix be

$$\boldsymbol{r} = \{a \cos t,\ a \sin t,\ bt\} ,$$

and the plane be the x, y-plane. Show that the required curve is an involute of the circle $x^2 + y^2 = a^2$.

4.11 Consider the point M which is a fixed distance c along the binormal to a curve with constant torsion τ. Determine the angle φ between the binormal to the given curve and the binormal to the curve which is the locus of the points M.

HINT:

Use the fact that the vector $\boldsymbol{r}' \times \boldsymbol{r}''$ is parallel to the binormal $\boldsymbol{b}$ to show that

$$\tan \varphi = \frac{c\tau^2}{\kappa}(1 + c^2\tau^2)^{-\frac{1}{2}} .$$

4.12 Consider the point M which is a fixed distance c along the tangent to the curve C with curvature κ and torsion τ. Find the curvature κ^* of the locus of M.

ANSWER:

$$(\kappa^*)^2(1 + c^2\kappa^2)^3 = c^2\kappa^2\tau^2(1 + c^2\kappa^2) + (\kappa + c\kappa' + c^2\kappa^3)^2 .$$

4.13 Repeat **4.12**, where M is now a fixed distance c along the binormal to the curve C.

ANSWER:

$$(\kappa^*)^2(1+c^2\tau^2)^3 = c^2\tau^4(1+c^2\tau^2)+(\kappa-c\tau'+c^2\kappa\tau^2)^2 .$$

4.14 Find the radius of curvature of the polar curve corresponding to a given curve $\boldsymbol{r}=\boldsymbol{r}(s)$ (see **3.8**).

HINT:

Let $\boldsymbol{r}^*=\boldsymbol{r}^*(s^*)$ be the equation of the polar curve. By differentiation with respect to s show that

$$\boldsymbol{t}^* = \boldsymbol{b}, \frac{\mathrm{d}s^*}{\mathrm{d}s} = \frac{\rho}{\sigma} + (\sigma\rho')' .$$

Use this result to show that

$$\rho^* = -[\rho+\sigma(\sigma\rho')'] .$$

4.15 Find the torsion of the polar curve corresponding to a given curve $\boldsymbol{r}=\boldsymbol{r}(s)$.

HINT:

Use the results of **4.14** to show that

$$\sigma^* = \frac{\rho}{\sigma}[\rho+\sigma(\sigma\rho')'] .$$

4.16 If the radius of spherical curvature of $\boldsymbol{r}=\boldsymbol{r}(s)$ is constant, prove that the curve either lies on a sphere or has constant curvature.

4.17 If a curve $\boldsymbol{r}=\boldsymbol{r}(s)$ lies on a sphere, show that ρ and σ are related by

$$\frac{\mathrm{d}}{\mathrm{d}s}(\sigma\rho') + \frac{\rho}{\sigma} = 0 .$$

4.18 Show that all curves, the osculating spheres of which have a fixed centre, lie on a sphere.

4.19 Let $\boldsymbol{r}=\boldsymbol{r}(s)$ be a curve for which the binormals at all points are parallel. Find the torsion for this curve.

HINT:

Let $\boldsymbol{t}^*$ and $\boldsymbol{t}$ denote the tangents to $\boldsymbol{r}=\boldsymbol{r}(s)$ at $s=s$ and $s=s+c$, where c is a constant parameter. Use the fact that the osculating planes at $s=s$ and $s=s+c$ must be parallel to show that $\tau \sin \alpha=0$, where α is the angle between $\boldsymbol{t}^*$ and $\boldsymbol{t}$.

4.20 The principal normals to a given curve $\boldsymbol{r}=\boldsymbol{r}(s)$ are also binormals to another curve $\boldsymbol{r}^*=\boldsymbol{r}^*(s^*)$. Prove that the distance between corresponding points of the curves is constant, and show that there is a relationship between the curvature and torsion for the curve $\boldsymbol{r}=\boldsymbol{r}(s)$.

HINT:

If d is the distance between corresponding points of the curves, then

$$\boldsymbol{r}^* = \boldsymbol{r} + d\boldsymbol{n} .$$

Use this result to show that d is a constant and that

$$\kappa = d(\kappa^2 + \tau^2) .$$

4.21 Two curves are called *Bertrand curves* when they have the same principal normals. Given two Bertrand curves, show that

(a) the distance between corresponding points is constant,

(b) the tangents at corresponding points are inclined at the same angle,

(c) there exists a linear relationship between the curvature and torsion of the curves.

4.22 Show that if the curvature and torsion of the curve $\boldsymbol{r}=\boldsymbol{r}(s)$ satisfy the linear relationship

$$P\kappa + Q\tau = R \qquad (P, Q, R=\text{const.} \neq 0) ,$$

then $\boldsymbol{r}=\boldsymbol{r}(s)$ belongs to a family of Bertrand curves.

HINT:

Consider the curves

$$\boldsymbol{r} = \boldsymbol{r}(s) \quad \text{and} \quad \boldsymbol{r}^*(s^*) = \boldsymbol{r}(s) + k\boldsymbol{n} ,$$

where $k = \text{const.}$ is defined by

$$(1 - k\kappa) \sin \alpha + k\tau \cos \alpha = 0 , \quad \alpha = \text{const.}$$

4.23 Find the curves $\boldsymbol{r} = \boldsymbol{r}(s)$ which have a constant ratio between their curvature and torsion.

HINT:

Consider the Serret–Frenet formulae

$$\boldsymbol{t}' = \kappa \boldsymbol{n} , \quad \boldsymbol{b}' = -\tau \boldsymbol{n} .$$

By elimination of $\boldsymbol{n}$, show that the required curves are cylindrical helices.

4.24 Show that the curve $\boldsymbol{r} = \{e^t, e^{-t}, \sqrt{2} \cdot t\}$ is a cylindrical helix, and determine its supporting cylinder.

HINT:

Use **4.23**, and show that

$$\kappa \boldsymbol{b} + \tau \boldsymbol{t} = \frac{\sqrt{2}}{(e^t + e^{-t})^2} \{-1, 1, 0\} .$$

4.25 Determine the curves which have constant curvature and torsion.

HINT:

Consider a circular helix and apply the Fundamental Theorem of space curves (see **3.6**).

4.26 Prove that a necessary and sufficient condition for the principal normals to a given curve $\boldsymbol{r} = \boldsymbol{r}(s)$ to be parallel to some fixed plane is that the given curve be a cylindrical helix.

HINT:

Use the fact that $\boldsymbol{t} \cdot \boldsymbol{e} = \text{const.}$ for a cylindrical helix, where $\boldsymbol{e}$ is some fixed vector.

4.27 Show that the curve

$$\boldsymbol{r} = \{4a\cos^3 t,\ 4a\sin^3 t,\ 3c\cos 2t\}$$

is a cylindrical helix.

HINT:
Use **4.26**.

4.28 Show that a necessary and sufficient condition for a curve $\boldsymbol{r}=\boldsymbol{r}(s)$ to be a helix is that

$$[\boldsymbol{r}'''', \boldsymbol{r}''', \boldsymbol{r}''] = 0\,.$$

HINT:
Using the Serret–Frenet formulae, show that

$$[\boldsymbol{r}'''', \boldsymbol{r}''', \boldsymbol{r}''] = -\kappa^5 \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\tau}{\kappa}\right).$$

4.29 Prove that a necessary and sufficient condition for a curve $\boldsymbol{r}=\boldsymbol{r}(x)$ to be a helix is that

$$[\boldsymbol{b}''', \boldsymbol{b}'', \boldsymbol{b}'] = 0\,.$$

4.30 Determine the curve of intersection between the x, y-plane and lines joining the point $P\{0, 0, c\}$ to points on the circular helix

$$\boldsymbol{r} = \{a\cos t,\ a\sin t,\ bt\}\,.$$

HINT:
Show that the required curve is a hyperbolic spiral with polar form

$$r(c - b\theta) = ac\,.$$

4.31 Determine the function $\varphi(t)$ so that the principal normals to the curve $\boldsymbol{r}=\{t, \sin t, \varphi(t)\}$ will be parallel to the y, z-plane.

HINT:
Show that

$$\varphi(t) = \frac{1}{k}\int (1-k^2\cos^2 t)^{\frac{1}{2}}\,dt\,,$$

where k is a constant.

4.32 Examine the parallel projection (not necessarily perpendicular to the x,y-plane) of the circular helix $\boldsymbol{r}=\{a\cos t,\ a\sin t,\ ht\}$ onto the x,y-plane.

HINT:

Let the unit vector $\boldsymbol{e}=\{e_1, e_2, e_3\}$ denote the direction of the projection. Show that the equation of the projection on the x,y-plane is

$$\boldsymbol{R} = \{a\cos t - bt,\ a\sin t - ct,\ 0\}\,, \tag{1}$$

where $b = \dfrac{e_1}{e_3}h$ and $c = \dfrac{e_2}{e_3}h$.

Introduce a rotation of axes defined by

$$x' = (bx+cy)/(b^2+c^2)^{\frac{1}{2}}, \quad y' = (cx-by)/(b^2+c^2)^{\frac{1}{2}}\ ;$$

and define θ by

$$\cos\theta = c/(b^2+c^2)^{\frac{1}{2}}\,.$$

Equation (1) becomes

$$x' = a\sin(\theta+t) - t(b^2+c^2)^{\frac{1}{2}}, \quad y' = a\cos(\theta+t)\,. \tag{2}$$

By introducing the further transformation

$$x'' = -x',\ y'' = (b^2+c^2)^{\frac{1}{2}} - y'\,,$$

equations (2) become

$$x'' = (b^2+c^2)^{\frac{1}{2}}t - a\sin(\theta+t), \quad y'' = (b^2+c^2)^{\frac{1}{2}} - a\cos(\theta+t)\,.$$

Depending on whether $(b^2+c^2)^{\frac{1}{2}}$ is less than, equal to, or greater than a, the curve is said to be an internal trochoid, a natural cycloid or an external trochoid (see [9]).

4.33 Let a plane α intersect the circular helix $\boldsymbol{r}=\{a\cos t,\ a\sin t,\ bt\}$. At all points of intersection construct the osculating planes to the helix and show that this family of planes has a common point of intersection.

HINT:

By (3.4.1), the equation of the osculating plane at a point t on the circular helix is

$$b(\sin t)\,x - b(\cos t)\,y + az - abt = 0\,,$$

where $\{x, y, z\}$ is an arbitrary point on the plane. Let the equation of the plane α be

$$x = A_1 + B_1 u + C_1 v, \quad y = A_2 + B_2 u + C_2 v, \quad z = A_3 + B_3 u + C_3 v,$$

where A_i, B_i and C_i $(i = 1, 2, 3)$ are constants. Show that the common point of intersection (X, Y, Z) is defined by

$$\begin{aligned} bA_2 X - bA_1 Y + a^2 Z &= a^2 A_3\,, \\ bB_2 X - bB_1 Y &= a^2 B_3\,, \\ bC_2 X - bC_1 Y &= a^2 C_3\,. \end{aligned}$$

4.34 Find the spherical indicatrix for the tangents to the curve $\boldsymbol{r} = \{2t - \sin 2t,\ -\cos 2t,\ 4 \sin t\}$.

ANSWER:

The indicatrix is the curve of intersection between the unit sphere and the cylinder $x^2 + y^2 - x = 0$.

4.35 Find the curvature κ^* and the torsion τ^* of the spherical indicatrix of

(a) the tangents to $\boldsymbol{r} = \boldsymbol{r}(s)$;

(b) the binormals to $\boldsymbol{r} = \boldsymbol{r}(s)$.

ANSWER:

(a) $\kappa^* = (\kappa^2 + \tau^2)^{\frac{1}{2}}/\kappa$, $\tau^* = (\kappa\tau' - \kappa'\tau)/[\kappa(\kappa^2 + \tau^2)]$;

(b) $\kappa^* = (\kappa^2 + \tau^2)^{\frac{1}{2}}/\tau$, $\tau^* = (\tau\kappa' - \kappa\tau')/\lfloor\tau(\kappa^2 + \tau^2)]$.

4.36 Consider the spherical indicatrix $\boldsymbol{t} = \boldsymbol{t}(t)$ of the tangents to the curve $\boldsymbol{r} = \boldsymbol{r}(s)$.

(a) Find the equation of $\boldsymbol{r} = \boldsymbol{r}(s)$ in terms of $\boldsymbol{t} = \boldsymbol{t}(t)$;

(b) Examine the special case when the indicatrix is a circle on the unit sphere.

(c) Find the general equation of a helix with given curvature, and with given torsion. Consider the special case when $\kappa = a \cos t$;

(d) Find the general equation of curves having constant curvature.

HINT:

(a) $\boldsymbol{r}(t) = \int_{t_0}^{t} c(u)\boldsymbol{t}(u)\mathrm{d}u$,

where t_0 is a constant and $c(u)$ is a scalar function.

(b) Let the circle be $\boldsymbol{r} = \{a \cos t,\ a \sin t,\ b\}$, then $c(t) = a/\kappa = b/\tau$, $\boldsymbol{n} = \{-\sin t,\ \cos t,\ 0\}$.

(c) $\boldsymbol{r} = a \int_{t_0}^{t} \frac{1}{\kappa(u)}\,\boldsymbol{t}(u)\mathrm{d}u$;

$\boldsymbol{r} = b \int_{t_0}^{t} \frac{1}{\tau(u)}\,\boldsymbol{t}(u)\mathrm{d}u$;

$\boldsymbol{r} = \left\{ at,\ -a \log(\cos t),\ \pm(1-a^2)^{\frac{1}{2}} \log \tan\left(\frac{t}{2} + \frac{\pi}{4}\right) \right\}$.

(d) Replace the parameter t by η (the arc length of the indicatrix). Then, from $\boldsymbol{r}' = \boldsymbol{t}$, we have

$$\boldsymbol{r}'' = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}\eta}\frac{1}{C(\eta)}.$$

It follows that the required equation is

$$\boldsymbol{r}(t) = \frac{1}{\kappa}\int_{t_0}^{t} \boldsymbol{t}(\eta)\mathrm{d}\eta\,.$$

4.37 Consider the spherical indicatrix $\boldsymbol{b} = \boldsymbol{b}(t)$ of the binormals of the curve $\boldsymbol{r} = \boldsymbol{r}(s)$.

(a) Find the equation of $\boldsymbol{r} = \boldsymbol{r}(s)$ in terms of $\boldsymbol{b} = \boldsymbol{b}(t)$.

(b) Examine the special case when the indicatrix is a circle on the unit sphere.

ANSWER:

(a) $\boldsymbol{r} = \int_{t_0}^{t} \frac{1}{\tau(u)}\left[\boldsymbol{b}(u) \times \frac{\mathrm{d}\boldsymbol{b}(u)}{\mathrm{d}u}\right]\mathrm{d}u$.

(b) Cylindrical helices.

4.38 Show that the involutes of a circular helix are plane curves in planes orthogonal to the axis of the helix. Also, show that these involutes are the involutes of the circular sections of the supporting cylinder.

4.39 Consider the curve $y=x^n$, $z=f(x)$, where n is a constant. Determine the function $f(x)$ such that the osculating planes at points M pass through points P which are the projections of M on the y-axis.

HINT:

Using equation (3.4.2) and the fact that the point $(0, y, 0)$ lies on the osculating plane, it follows that

$$\begin{vmatrix} x & 0 & f(x) \\ 1 & x & f'(x) \\ 0 & n-1 & f''(x) \end{vmatrix} = 0 .$$

Solving the differential equation defined by the last expression, one finds that

$$f(x) = ax^{n-1} + bx ,$$

where a and b are arbitrary constants.

4.40 Consider the curve

$$\boldsymbol{r} = \{c_1 t, c_2 t^2, c_3 t^3\} .$$

(a) Find the plane passing through the three points corresponding to $t=t_1$, t_2 and t_3.

(b) Hence, find the equation of the osculating plane at any point of the curve.

HINT:

(a) Let the equation of the plane be

$$a_1 x + a_2 y + a_3 z + a_4 = 0 .$$

Using the relationships between the roots and the coefficients of a cubic equation, the equation of the plane is found to be

$$c_2c_3(t_1t_2+t_2t_3+t_3t_1)x-c_1c_3(t_1+t_2+t_3)y+ \\ +c_1c_2z-c_1c_2c_3t_1t_2t_3=0\,. \qquad (1)$$

(b) Using (1), the equation of the osculating plane at t is found to be

$$3c_2c_3t^2x-3c_1c_3ty+c_1c_2z-c_1c_2c_3t^3=0\,. \qquad (2)$$

Result (2) can be checked by using (3.4.1) to determine the required osculating plane.

4.41 Consider the curve C, $\boldsymbol{r}=\{t, t^2, t^3\}$, and let the point $M_0=\{x_0, y_0, z_0\}$ be any point not on this curve. Show that, in general, there are three points M_1, M_2 and M_3 lying on C such that the tangent planes to C at M_1, M_2 and M_3 intersect at M_0. Prove that the points M_0, M_1, M_2 and M_3 all lie on the same plane.

HINT:

The equation of the osculating plane passing through M_0 is

$$t^3-3x_0t^2+3y_0t-z_0=0\,.$$

The three points M_1, M_2 and M_3 will be defined when M_0 is such that this cubic in t has real roots. The four points M_0, M_1, M_2 and M_3 lie on the plane $3x_0y-3y_0x+z-z_0=0$.

4.42 Consider the two curves $\boldsymbol{r}_1=\boldsymbol{r}_1(s_1)$ and $\boldsymbol{r}_2=\boldsymbol{r}_2(s_2)$. Prove that the shortest distance between the given curves is the chord which is orthogonal to both curves.

CHAPTER 5

ELEMENTS OF THE THEORY OF SURFACES

5.1 THE CONCEPT OF SURFACE

Let the components of the position vector $\boldsymbol{r}$ be single-valued and continuously differentiable functions of two variable parameters u^1 and u^2, where

$$\boldsymbol{r}(u^1, u^2) = \{x^1(u^1, u^2),\ x^2(u^1, u^2),\ x^3(u^1, u^2)\}\ . \qquad (5.1.1)$$

If in some region of the variables u^1 and u^2

$$\boldsymbol{r}_1 \times \boldsymbol{r}_2 \neq 0, \text{ where } \boldsymbol{r}_i = \partial \boldsymbol{r}/\partial u^i \qquad (i=1, 2)\ ,$$

then the end point of $\boldsymbol{r}$ defines a point set which is called a *continuously differentiable surface.*

NOTE: In what follows the notation $\boldsymbol{r}_i (i = 1, 2)$, or $\boldsymbol{r}_u = \partial \boldsymbol{r}/\partial u$, $\boldsymbol{r}_v = \partial \boldsymbol{r}/\partial v$, where $\boldsymbol{r} = \boldsymbol{r}(u, v)$, will be used.

The parametric form for the surface (5.1.1) is

$$x^i = x^i(u^1, u^2) \qquad (i=1, 2, 3)\ . \qquad (5.1.2)$$

Eliminating the parameters u^1 and u^2 from (5.1.2), one obtains the implicit form

$$F(x^1, x^2, x^3) = 0 \qquad (5.1.3)$$

which is satisfied for all points $\{x^1, x^2, x^3\}$ of the surface (5.1.2).

5.2 THE TANGENT PLANE AND THE NORMAL

The curves $u^1 = \text{const.}$ and $u^2 = \text{const.}$ on the surface (5.1.2) are called *coordinate curves.* The vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are parallel to the tangents of the curves $u_2 = \text{const.}$ and $u_1 = \text{const.}$, respectively.

Substitution of $u^1 = u^1(t)$, $u^2 = u^2(t)$ in (5.1.1) gives $\boldsymbol{r} = \boldsymbol{r}(u^1(t), u^2(t))$, which is the equation of a curve on the surface (5.1.1). The vector

$$\dot{\boldsymbol{r}} = \boldsymbol{r}_1 \dot{u}^1 + \boldsymbol{r}_2 \dot{u}^2 \,, \quad \text{where } \dot{u}^i = \frac{\mathrm{d}u^i}{\mathrm{d}t} \qquad (i = 1, 2),$$

coincides with the tangent to this surface curve. The tangents to all surface curves, which pass through a fixed point (u^1, u^2) of the surface, lie in the plane of $\boldsymbol{r}_1(u^1, u^2)$ and $\boldsymbol{r}_2(u^1, u^2)$. This plane is called the *tangent plane* to the surface at the point (u^1, u^2). It has the equation

$$[\boldsymbol{R} - \boldsymbol{r}(u^1, u^2),\ \boldsymbol{r}_1(u^1, u^2),\ \boldsymbol{r}_2(u^1, u^2)] = 0 \,,$$

where $\boldsymbol{R}$ is a variable position vector defining points in the plane.

The straight line through a fixed point (u^1, u^2) of the surface with the direction $(\boldsymbol{r}_1 \times \boldsymbol{r}_2)$ is called the *normal to the surface* or *surface normal* at the point (u^1, u^2). The equation of this normal is

$$\mu \boldsymbol{N} = \boldsymbol{r}(u^1, u^2) + \lambda(\boldsymbol{r}_1 \times \boldsymbol{r}_2) \,,$$

where λ is a variable parameter defining points on the normal, μ the length of the normal and $\boldsymbol{N}$ the unit surface normal.

When the equation of a surface has the form (5.1.3), the equations to the tangent plane and the normal at the point $\{x^1, x^2, x^3\}$, respectively, are given by

$$(X^1 - x^1) F_{x^1} + (X^2 - x^2) F_{x^2} + (X^3 - x^3) F_{x^3} = 0 \,,$$

and

$$\frac{X^1 - x^1}{F_{x^1}} = \frac{X^2 - x^2}{F_{x^2}} = \frac{X^3 - x^3}{F_{x^3}} \,,$$

where $\boldsymbol{R} = \{X^1, X^2, X^3\}$ denotes a variable point on either the tangent plane or the normal.

The *tangent surface* of a curve C, $\boldsymbol{r} = \boldsymbol{r}(s)$, is the surface generated by lines tangent to C. It has the equation

$$\boldsymbol{r}(s, \lambda) = \boldsymbol{r}(s) + \lambda \boldsymbol{t}(s) \,,$$

where λ is a parameter and $\boldsymbol{t}$ is the tangent to C.

5.3 THE ENVELOPE

The surface which is tangent to every member of a family of surfaces is called its *envelope*. Such a family is defined by the equation

$$\boldsymbol{r} = \boldsymbol{r}(u^1, u^2, \lambda), \tag{5.3.1}$$

where λ is the family parameter. The envelope consists of those points which satisfy (5.3.1) and the equation

$$[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_\lambda] = 0,$$

under the assumption that this last equation does not define a member of the family (5.3.1).

If the family of surfaces is defined by the equation

$$F(x, y, z, \lambda) = 0,$$

where the form of F is such that $\partial F/\partial x$, $\partial F/\partial y$ and $\partial F/\partial z$, are non-zero, then its envelope is determined by the equations

$$F(x, y, z, \lambda) = 0, \quad \frac{\partial}{\partial \lambda} F(x, y, z, \lambda) = 0. \tag{5.3.2}$$

For a fixed value of λ, equations (5.3.2) determine a curve which is called a *characteristic of the family*. Along this curve the envelope will touch the surface of the family to which λ corresponds. The envelope of all the characteristics is called the *edge of regression of the family*. It is defined by the equations

$$F = 0, \quad \frac{\partial F}{\partial \lambda} = 0, \quad \frac{\partial^2 F}{\partial \lambda^2} = 0.$$

5.4 DEVELOPABLE SURFACES

If a straight line moves in any direction along a curve, a *ruled surface* is generated. This moving line is called a *generator*. The tangent plane to a ruled surface sometimes coincides with the generator. In the special case when every tangent plane to a surface is tangent to a generator, the surface is called a *developable surface*. This class of surfaces contains as well as planes, cylinders and cones, the tangent surfaces of any

curve. In the latter case the curve is the *edge of regression* of the developable surface (see [4]).

The ruled surface

$$\bar{\boldsymbol{r}}(s, u) = \boldsymbol{r}(s) + u\boldsymbol{e}(s), \tag{5.4.1}$$

where $\boldsymbol{e} = \boldsymbol{e}(s)$ is a unit vector, is developable if

$$[\boldsymbol{r}_s, \boldsymbol{e}, \boldsymbol{e}'] = 0.$$

When (5.4.1) is developable, its edge of regression is defined by (5.4.1) and

$$|\bar{\boldsymbol{r}}_s \times \bar{\boldsymbol{r}}_u| = 0.$$

When the surface $\bar{\boldsymbol{r}}(s, u)$ has the form

$$z = F(x, y),$$

the condition for developability becomes

$$\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} = \left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2.$$

5.5 THE INVOLUTES AND EVOLUTES OF A CURVE

The curves which lie on the tangent surface of a space curve C and which intersect at right angles the tangents to C, are called the *involutes* of C. For the curve $\boldsymbol{r} = \boldsymbol{r}(s)$, involutes are defined by

$$\boldsymbol{R} = \boldsymbol{r}(s) + (c - s)\boldsymbol{r}'(s),$$

where $c = \text{const}$.

A curve, which has as its involute a given curve C, is called an *evolute* of C. For the curve $\boldsymbol{r} = \boldsymbol{r}(s)$, the evolute is defined by

$$\boldsymbol{R} = \boldsymbol{r}(s) + \rho\boldsymbol{n}(s) + \rho\cot\theta\,\boldsymbol{b}(s),$$

where $\theta = \int \tau\,ds$. This equation defines a one-parameter family of evolutes.

5.6 LENGTH

The differential of arc length ds of a curve $u^i = u^i(t)$ $(i = 1, 2)$ on the surface $\boldsymbol{r} = \boldsymbol{r}(u^1, u^2)$ is defined by

$$(\mathrm{d}s)^2 = g_{11}(\mathrm{d}u^1)^2 + 2g_{12}\,\mathrm{d}u^1\mathrm{d}u^2 + g_{22}(\mathrm{d}u^2)^2 ,$$

where $g_{ik} = \boldsymbol{r}_i \cdot \boldsymbol{r}_k$. This expression for $(\mathrm{d}s)^2$ is called the *first fundamental form of the surface.* Using the summation convention, this expression can be written in the form

$$(\mathrm{d}s)^2 = g_{ik}\,\mathrm{d}u^i\mathrm{d}u^k \qquad (i, k = 1, 2) .$$

Writing u, v instead of u^1, u^2 and E, F, G, instead of g_{11}, g_{12}, g_{22}, one obtains the classical expression for $\mathrm{d}s$

$$(\mathrm{d}s)^2 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 .$$

For the surface $z = F(x, y)$, one has $E = 1 + p^2$, $F = pq$, $G = 1 + q^2$, where $p = \partial z/\partial x$, $q = \partial z/\partial y$ are Monge's abbreviations for the first derivatives of z.

The *arc length* of the curve $u^i = u^i(t)$ $(i = 1, 2)$ between the points t_1 and t_2 is

$$s = \int_{t_1}^{t_2} (g_{ik}\dot{u}^i\dot{u}^k)^{\frac{1}{2}}\,\mathrm{d}t .$$

5.7 ANGLE

On a surface with the first fundamental form $g_{ik}\mathrm{d}u^i\mathrm{d}u^k$, consider the two curves $u^i = u^i(t)$ and $v^i = v^i(t)$ $(i = 1, 2)$. If they intersect at a point, then the tangents to these two curves at the point of intersection define an angle α which is defined by

$$\cos\alpha = g_{ik}\mathrm{d}u^i\mathrm{d}v^k / [(g_{ik}\mathrm{d}u^i\mathrm{d}u^k)^{\frac{1}{2}}(g_{ik}\mathrm{d}v^i\mathrm{d}v^k)^{\frac{1}{2}}] ,$$

or

$$\tan\alpha = g^{\frac{1}{2}}(\mathrm{d}u^1\mathrm{d}v^2 - \mathrm{d}u^2\mathrm{d}v^1)/(g_{ik}\mathrm{d}u^i\mathrm{d}v^k) ,$$

where

$$g = g_{11}g_{22} - (g_{12})^2 .$$

Two families of intersection curves (C_1, C_2) on a surface are said to form a *net*, if at any point of the surface one and only one curve from each family passes through the point. If on a surface a net (C_1, C_2) is defined such that the angle between the curves at any point is constant, the net is called *isogonal.* The curves $C_1(C_2)$ are called the *isogonal*

trajectories of C_2 (C_1). When the angle is $\frac{1}{2}\pi$, the net is called *orthogonal* and C_1 (C_2) are the *orthogonal trajectories* of C_2 (C_1).

The coordinate curves $u^1 = \text{const.}$ and $u^2 = \text{const.}$ on a surface form a net. If the coordinate curves are such that the first fundamental form of the surface is

$$(\mathrm{d}s)^2 = \lambda(u^1, u^2)\{(\mathrm{d}u^1)^2 + (\mathrm{d}u^2)^2\},$$

then the net is called an *isometric* (or *isothermic*) net.

NOTE. The *area* of a region $B(u^1, u^2)$ of the surface $\boldsymbol{r} = \boldsymbol{r}(u^1, u^2)$ is defined by the double integral

$$A = \iint_{B(u^1, u^2)} g^{\frac{1}{2}}\,\mathrm{d}u^1\,\mathrm{d}u^2,$$

where $$g = [g_{11}g_{22} - (g_{12})^2] = [|\boldsymbol{r}_1 \times \boldsymbol{r}_2|]^2.$$

5.8 THE SECOND FUNDAMENTAL FORM OF A SURFACE AND THE CURVATURE OF A SURFACE CURVE

Since the normal $\boldsymbol{N}$ to the surface $\boldsymbol{r} = \boldsymbol{r}(u^1, u^2)$ has the direction of the vector $\boldsymbol{r}_1 \times \boldsymbol{r}_2$, one has

$$\boldsymbol{N} = g^{-\frac{1}{2}}(\boldsymbol{r}_1 \times \boldsymbol{r}_2).$$

On the given surface, consider the curve $u^i = u^i(t)$ $(i = 1, 2)$. The radius of curvature ρ of this curve at the point P and the angle θ between the normals $\boldsymbol{n}$ and $\boldsymbol{N}$ at P are related by

$$\frac{\cos\theta}{\rho} = \frac{h_{ik}\mathrm{d}u^i\mathrm{d}u^k}{g_{ik}\mathrm{d}u^i\mathrm{d}u^k}, \tag{5.8.1}$$

where $h_{ik} = [\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{ik}]g^{-\frac{1}{2}}$, $\boldsymbol{r}_{ik} = \partial^2\boldsymbol{r}/\partial u^i\,\partial u^k$. In fact, $h_{ik} = -\boldsymbol{r}_i \cdot \boldsymbol{N}_k$. In the classical literature, the quantities h_{11}, h_{12}, h_{22} are denoted by L, M, N.

For a surface $z = F(x, y)$,

$$L = \frac{r}{H}, \quad M = \frac{s}{H}, \quad N = \frac{t}{H},$$

where $H^2 = EG - F^2$, and $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\partial y$, $t = \partial^2 z/\partial y^2$ are Monge's abbreviations.

The quadratic differential form $h_{ik}du^i du^k$ is called the *second fundamental form* of the surface.

5.9 THE THEOREM OF MEUSNIER. THE NORMAL SECTION

The intersection of a surface with a plane which passes through the surface normal at a point P of the surface is called a *normal section* of the surface at P. The curvature of the normal section at P will be denoted by κ_n and called the *normal curvature*. For all surface curves which pass through and have the same tangent at a point P, one has the Meusnier Theorem (see [8])

$$\kappa \cos\theta = \kappa_n ,$$

where θ is the angle between the normals $\boldsymbol{n}$ and $\boldsymbol{N}$, and κ is the curvature of the curve at P. Hence, κ_n can be determined by (5.8.1).

5.10 THE INDICATRIX OF DUPIN AND THE FORMULA OF EULER

A point P at which

$$\frac{g_{11}}{h_{11}} = \frac{g_{12}}{h_{12}} = \frac{g_{22}}{h_{22}}$$

is called an *umbilical* point. At such a point, the curvature is the same for all normal sections.

If this is not the case, then the normal curvature κ_n at the point P has two extreme values which determine two tangents at P. The stationary values of κ_n are called the *principal curvatures* of the surface at P and the corresponding tangents the *principal directions* at P. The principal curvatures are denoted by κ_1 and κ_2 and are the roots of the equation

$$g\kappa^2 - (g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11})\kappa + h = 0 ,$$

where $h = h_{11}h_{22} - (h_{12})^2$.

It can be shown that a plane parallel to the tangent plane through P intersects the surface near P in a conic. This conic, called *Dupin's indicatrix*, can be an ellipse, hyperbola, or a pair of straight lines. Depending on whether Dupin's indicatrix is an ellipse, hyperbola or

a pair of straight lines, the surface at P is said to have elliptic, hyperbolic or parabolic curvature.

Let ϕ denote the angle between any direction at P and the principal direction at P corresponding to κ_1. Then, the normal curvature κ_n in the given direction at P is defined by Euler's formula

$$\kappa_n = \kappa_1 \cos^2\varphi + \kappa_2 \sin^2\varphi \,.$$

5.11 THE GAUSSIAN AND AVERAGE CURVATURE

The product, $K = \kappa_1\kappa_2$, of the principal curvatures at a point P of a surface is called the *Gaussian curvature* of the surface at P, and the arithmetic mean, $H = \frac{1}{2}(\kappa_1 + \kappa_2)$, of the principal curvatures is called the *average curvature* of the surface at P. It follows from **5.10** that

$$K = h/g \,,$$

$$H = (g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11})/(2g) \,.$$

An important result regarding the Gaussian curvature of a developable surface is: *A necessary and sufficient condition for a surface to be a developable surface is that the Gaussian curvature at all points of the surface be zero.*

5.12 THE THEOREM OF GAUSS AND CHRISTOFFEL SYMBOLS

Gauss has shown that the Gaussian curvature K can be expressed in terms of g_{ik} and its derivatives (the Theorem of Gauss):

$$g^{\frac{1}{2}}K = \frac{\partial}{\partial u^2}\left(\frac{g^{\frac{1}{2}}}{g_{11}}\Gamma^2_{11}\right) - \frac{\partial}{\partial u^1}\left(\frac{g^{\frac{1}{2}}}{g_{11}}\Gamma^2_{12}\right) \,,$$

where

$$\Gamma^l_{ik} = g^{lm}\Gamma_{ik,m} \qquad \Gamma_{ik,m} = \tfrac{1}{2}\left(\frac{\partial g_{im}}{\partial u^k} + \frac{\partial g_{km}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^m}\right) \,,$$

and g^{lm} is defined by

$$g^{11} = \frac{g_{22}}{g}, \quad g^{12} = -\frac{g_{12}}{g}, \quad g^{22} = \frac{g_{11}}{g}.$$

The symbols $\Gamma_{ik,m}$ and Γ^l_{ik} are called the Christoffel symbols of the first and second kinds, respectively.

5.13 THE CONJUGATE DIRECTIONS AND ASYMPTOTIC LINES

The tangents to the surface curves $u^i = u^i(t)$ and $v^i = v^i(t)$ $(i = 1, 2)$ at their point of intersection are called *conjugate directions*, if

$$h_{ik} \mathrm{d}u^i \mathrm{d}v^k = 0 .$$

Geometrically this concept is characterised by the following theorem: *the generators (characteristic lines) of a developable surface, which is the envelope of the one parameter family of tangent planes of a curve C of the surface, are conjugate directions with respect to the tangents of C* (see [8]).

Two families of intersecting curves, which form a net on a surface S, define a *conjugate net* (or conjugate family of curves), if the tangents to the curves of different families define conjugate directions at each point of the surface S.

The tangent to a surface curve defines an *asymptotic direction* if its direction is self-conjugate. A curve, the tangent of which at every point defines an asymptotic direction, is called an *asymptotic line* of the surface. It follows that the equation of an asymptotic line is

$$h_{ik} \mathrm{d}u^i \mathrm{d}u^k = 0 .$$

If the curvature at a point P is elliptic, parabolic or hyperbolic, then there exist no, one or two asymptotic lines which pass through P.

An important result is the Beltrami–Enneper Theorem: *the torsion τ of an asymptotic line at a point P equals the negative value of the Gaussian curvature K at P.*

5.14 LINES OF CURVATURE

A surface curve, is called a line of curvature, if its tangents coincide at every point with a principal direction. A line of curvature is defined by the differential equation:

$$\begin{vmatrix} (\mathrm{d}u^2)^2 & -\mathrm{d}u^1\,\mathrm{d}u^2 & (\mathrm{d}u^1)^2 \\ g_{11} & g_{12} & g_{22} \\ h_{11} & h_{12} & h_{22} \end{vmatrix} = 0 .$$

A curve on a surface S is a line of curvature if and only if the formula of Rodrigue,

$$\kappa_n \mathrm{d}\boldsymbol{r} + \mathrm{d}\boldsymbol{N} = 0$$

holds for all points of the curve.

5.15 GEODESIC CURVATURE AND GEODESICS

A curve drawn on a surface so that its osculating plane at any point P contains the normal to the surface at P is a *geodesic*. It follows that all straight lines on a surface are geodesics, and that on a curved geodesic the principal normal coincides at all points with the normal to the surface.

For any surface curve $\boldsymbol{r}=\boldsymbol{r}(s)$ the curvature vector at a point P is given by $\boldsymbol{r}_c=\kappa\boldsymbol{n}$, where κ is the curvature and $\boldsymbol{n}$ is the principal normal. In terms of $\boldsymbol{N}$, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, this can be written as

$$\boldsymbol{r}_c = \kappa_n \boldsymbol{N} + \lambda \boldsymbol{r}_1 + \mu \boldsymbol{r}_2 ,$$

where κ_n is the normal curvature. The vector $\lambda\boldsymbol{r}_1+\mu\boldsymbol{r}_2$ is zero for a geodesic, since $\boldsymbol{n}$ and $\boldsymbol{N}$ coincide. This vector is called the *geodesic curvature vector.*

The geodesic curvature, κ_g, of any curve is defined as the magnitude of the geodesic curvature vector, with sign attached; it is given by

$$\kappa_g = [\boldsymbol{N}, \boldsymbol{r}'(s), \boldsymbol{r}''(s)] .$$

NOTE 1: Geodesic lines are the shortest lines which lie between two given points on a given surface (or they define a stationary value for the arc lengths of curves on a surface between two given points).

NOTE 2: The geodesic curvature vector of any curve is orthogonal to the curve. Further, $\kappa_g=\kappa \sin\theta$, where κ and θ are defined in **5.9**.

The differential equations defining the goedesics are

$$\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j = 0 \qquad (k=1, 2),$$

where the dots denote differentiation with respect to arc length (this will be the case throughout the remainder of this sections).

These equations can also be expressed in the following forms

$$g_{li}\ddot{u}^i + \Gamma_{ij,k}\dot{u}^j\dot{u}^k = 0 \qquad (l=1, 2),$$

or (5.15.1)

$$2\frac{\mathrm{d}}{\mathrm{d}s}(g_{li}\dot{u}^i) = \frac{\partial g_{ik}}{\partial u^l}\dot{u}^i\dot{u}^k \qquad (l=1, 2).$$

5.16 THE MAPPING OF SURFACES

Consider two surfaces S and S', defined by the equations $\boldsymbol{r}=\boldsymbol{r}(u^1, u^2)$ and $\boldsymbol{r}=\boldsymbol{r}(v^1, v^2)$, respectively. Corresponding to the points (v^1, v^2) on S', let the points (u^1, u^2) on S be determined by the single-valued functions

$$u^1 = \phi(v^1, v^2), \quad u^2 = \psi(v^1, v^2). \tag{5.16.1}$$

If the Jacobian of (5.16.1)

$$\frac{\partial(u^1, u^2)}{\partial(v^1, v^2)} \neq 0,$$

and the correspondence between the points (u^1, u^2) and (v^1, v^2) is one to one, then a *regular mapping* of S' onto S is defined. Because of its definition, the mapping will have a uniquely determined inverse.

Using (5.16.1), the equation of S can be written

$$\boldsymbol{r}(u^1, u^2) = \boldsymbol{r}(\phi(v^1, v^2), \psi(v^1, v^2)) = \boldsymbol{r}^*(v^1, v^2).$$

Thus, corresponding points on S and S' can be determined by the same parameter values v^1 and v^2.

For this reason, it is assumed that the surfaces S and S' are determined by the equations

$$\boldsymbol{r} = \boldsymbol{r}(u^1, u^2) \quad \text{and} \quad \boldsymbol{r} = \bar{\boldsymbol{r}}(v^1, v^2) \tag{5.16.2}$$

and the mappings by

$$u^1 = v^1(u^1, u^2) \quad \text{and} \quad u^2 = v^2(u^1, u^2). \tag{5.16.3}$$

Isometric mappings. A mapping between S and S' is called *isometric*, if the correspondence between the points of S and S' is such that corresponding arcs of curves have the same length. This correspondence is called *isometry*.

The mapping (5.16.3) of surfaces (5.16.2) is isometric if and only if, for all possible points (u^1, u^2),

$$g_{ik} = \bar{g}_{ik} .$$

In general, two arbitrarily given surfaces are not isometrically related. An isometric mapping preserves not only length, but also angle and area.

Conformal mappings. A mapping between S and S' is called *conformal*, if it preserves angles, that is, if the angle between any two curves at an arbitrary point P on S is equal to the angle between the corresponding curves on S'.

The mapping (5.16.3) of surfaces (5.16.2) is conformal if and only if, for all possible points (u^1, u^2).

$$\frac{g_{11}}{\bar{g}_{11}} = \frac{g_{12}}{\bar{g}_{12}} = \frac{g_{22}}{\bar{g}_{22}} .$$

Every point on a surface S has a neighbourhood which can be mapped conformally onto a neighbourhood of a point on any other surface.

Area preserving mappings. A mapping is called *area preserving*, if the surface area of corresponding region on S and S' are the same. The mapping (5.16.3) preserves area if and only if, for all points (u^1, u^2),

$$g = \bar{g} .$$

If $g = c\bar{g}$, $c = \text{const.}$, the mapping is said to be *generalised area preserving.*

CHAPTER 6

PROBLEMS RELATING TO THE PROPERTIES OF SURFACES

NOTE. Throughout this chapter u^1 and u^2 will often be replaced by u and v; and k will denote a non-zero constant.

6.1 SURFACES

6.1.1 Consider the equation

$$\boldsymbol{r} = \{u \cos v,\ u \sin v,\ (k^2 - u^2)^{\frac{1}{2}}\} . \tag{1}$$

(a) What surface is represented by this equation?
(b) Find the coordinate curves $u = \text{const.}$ and $v = \text{const.}$
(c) Give a geometrical interpretation of the parameters u and v.

ANSWER:

(a) From (1) it is seen that

$$x^2 + y^2 + z^2 = k^2 , \qquad z \geqslant 0 , \tag{2}$$

which is the equation of the hemisphere of radius k with center at the origin lying above the x, y-plane.

(b) Since $y = x \tan v$, the u-curves (i.e., the curves with $v = \text{const.}$) are sections between the hemisphere (2) and the planes containing the z-axis and the lines $y = x \tan v$. These curves are called *meridians* (or *lines of longitude*) of the hemisphere.

When $u = \text{const.}$, $z = (k^2 - u^2)^{\frac{1}{2}} = \text{const.}$, i.e. the v-curves are the circles obtained as sections of the hemisphere by planes parallel to the x, y-plane. These curves are called *parallels* (or *lines of latitude*) of the hemisphere.

(c) The distance of the point (u, v) from the z-axis is given by u, while v defines the latitude of this point with respect to some fixed directions in the x, y-plane.

6.1.2 Repeat **6.1.1** for the equation

$$\boldsymbol{r} = \{u \cos v,\ u \sin v,\ kv\}\,, \qquad k>0\,.$$

ANSWER:

(a) The surface is the helicoid which is generated by rotating a straight line, parallel to the x,y-plane, about the z-axis so that it remains in contact with a given circular helix.

(b) The curves $u=$const. are circular helices of radius u and pitch $2k\pi$ (see [7]). The curves $v=$const. are the straight lines defined by the section of the helicoid with the planes containing the z-axis and the line $y=x \tan v$.

(c) u is the distance of a point on the helicoid from the z-axis, and v is the angle of rotation about the z-axis, where the position $v=0$ corresponds to the positive x-axis.

6.1.3 Repeat **6.1.1** for the equation

$$\boldsymbol{r} = \{k \cos u \cos v,\ k \cos u \sin v,\ k \sin u\}, \qquad k>0\,.$$

6.1.4 Repeat **6.1.1** for the following surface equations:

(a) $x_i = k_i u + b_i v + c_i \qquad (i=1, 2, 3)$;

(b) $\boldsymbol{r} = \{k \cos u,\ k \sin u,\ v\}$;

(c) $\boldsymbol{r} = \{k \cos u,\ k \sin u,\ b(u+v)\}$;

(d) $\boldsymbol{r} = \{f(u), g(u), v\}$, where $f(u)$ and $g(u)$ are arbitrary functions of u;

(e) $\boldsymbol{r} = \{u, v, kuv\}$;

(f) $\boldsymbol{r} = \{k \cos u \cos v,\ b \cos u \sin v,\ c \sin u\}$,

where the k_i, b_i and c_i $(i=1, 2, 3)$ denote constants and the k, b and c non-zero constants.

HINT:

(a) Three cases: (i) If $\boldsymbol{k} = \{k_1, k_2, k_3\}$ and $\boldsymbol{b} = \{b_1, b_2, b_3\}$ are not parallel, then the surface is a plane in which $\boldsymbol{k}$ and $\boldsymbol{b}$ define two independent directions with origin at $C(c_1, c_2, c_3)$; (ii) If $\boldsymbol{k}$ and $\boldsymbol{b}$ are parallel and non-zero, then the equations determine a line parallel to $\boldsymbol{k}$ through the point C; (iii) If $\boldsymbol{k}=\boldsymbol{b}=\boldsymbol{0}$, then the equations determine the point C.

(b) A circular cylinder of radius k.

(c) The same as (b) except that the curves $v=$ const. are now circular helices.

(d) A general cylinder the cross-section of which is defined by $x=f(u)$ and $y=g(u)$.

(e) A hyperbolic paraboloid.

(f) An ellipsoid.

6.1.5 What surface is generated by the principal normals of a circular helix?

ANSWER:

A section of a right helicoid (see [7]).

6.1.6 Find the equation of the surface which is generated by moving a straight line so that it cuts a given straight line at a constant angle α. Examine the special cases when $\alpha=\frac{1}{2}\pi$.

HINT:

Let the fixed line be the z-axis and the generator be

$$\frac{x}{1}=\frac{y}{t}=\frac{z-k_1(t)}{k_2(t)}$$

where t is a parameter. By evaluating $\cos\alpha$, show that

$$k_2(t)=[1+t^2]^{\frac{1}{2}}\cot\alpha\,.$$

Use this result to show that required equation is

$$z=k_1\left(\frac{y}{x}\right)+[x^2+y^2]^{\frac{1}{2}}\cot\alpha\,.$$

6.2 THE TANGENT PLANE AND THE NORMAL

6.2.1 Consider the curve $\boldsymbol{r}=\boldsymbol{r}(s)$.

(a) What is the surface

$$\bar{\boldsymbol{r}}(s,u)=\boldsymbol{r}(s)+u\boldsymbol{t}(s)\,,$$

where $\boldsymbol{t}$ is the unit tangent of $\boldsymbol{r}=\boldsymbol{r}(s)$ and u is a parameter independent of s?

(b) Show that the tangent planes to this surface are independent of u.

HINT:

(b) Show that the equation of the tangent plane to this surface has the form

$$(\boldsymbol{R}-\boldsymbol{r})\cdot\boldsymbol{b}=0\,,$$

where $\boldsymbol{R}$ is the position vector defining points in the plane. Note that it follows then that tangent planes at all points of a generator of a tangent surface coincide (see **5.4**).

6.2.2 Find the section of the tangent surface of the curve (Viviani's curve)

$$\boldsymbol{r}=\left\{k(1+\cos u),\ k\sin u,\ 2k\sin\frac{u}{2}\right\}$$

with the x, y-plane.

ANSWER:

The cissoid: $xy^2+(x-2k)^3=0$.

6.2.3 Consider the equation $\boldsymbol{r}=\{u\cos v,\ u\sin v,\ f(v)\}$ where $f(v)$ is an arbitrary function of v.

(a) What surface is represented by this equation? Express this equation in the form $z=F(x, y)$.

(b) Find the tangent plane to this surface.

ANSWER:

The surface is a conoid (see [7] and **6.1.6**).

6.2.4 For the surface $\boldsymbol{r}=\{(k+u)\cos v,\ (k-u)\sin v, u\}$ find the locus of the points at which the tangent planes to the surface are parallel to the z-axis. Show that the projection of this curve on the x, y-plane is a closed curve.

ANSWER:

Using the equation of the tangent plane to the given surface, one finds that the planes are parallel to the z-axis if $u=k\cos 2v$. Hence the required curve is

$$\boldsymbol{r}=\{2k\cos^3 v,\ 2k\sin^3 v,\ k\cos 2v\}\,.$$

6.2.5 Consider the equation $r=\{u\cos v,\ u\sin v,\ f(u)\}$, where $f(u)$ is an arbitrary function of u.

(a) What surface is represented by this equation?

(b) Express this equation in the form $z=F(x, y)$.

(c) What is the equation of the circular cone the rectilinear generators of which make an angle α with its axis?

ANSWER:

(a) A surface of revolution formed by rotating the curve $z=f(x)$, $y=0$, about the z-axis.

(b) $z=f([x^2+y^2]^{\frac{1}{2}})$.

(c) If the axis of the cone is the z-axis and the vertex of the cone is at the origin, then the required equation is

$$x^2+y^2=z^2\tan^2\alpha\,.$$

6.2.6 Find the surface generated by the surface normals at all points of one of the rectilinear generators of a conoid.

HINT:

The equation of the required surface is

$$\bar{r}(u,\lambda)=r(u, v_0)+[\lambda(r_u\times r_v)]_{v=v_0}\,,$$

where $r=r(u, v)$ is the general equation of the conoid (see **6.2.3**) and $v=v_0$ determines a particular generator of the conoid. Elimination of λ and u from this equation shows that the surface is a hyperbolic paraboloid or a plane depending on whether $[\mathrm{d}f/\mathrm{d}v]_{v=v_0}$ is non-zero or zero (see **6.2.3**).

6.2.7 Consider the surface

$$r(u, v)=\{e^{hv}f(u)\cos(u+v),\ e^{hv}f(u)\sin(u+v),\ e^{hv}\varphi(u)\}\,,$$

where $f(u)$ and $\varphi(u)$ are given functions of u, and h is a constant. Show that the curves $u=$const. lie on the cones

$$x^2+y^2-\frac{f^2}{\varphi^2}z^2=0\,,$$

and determine the angles between the $u=$const.-curves and coordinate curves of the cone.

HINT:

Show that the curves $u=$const. are loxodromes; i.e., they cut the coordinate curves of the cones at constant angles.

6.2.8 Find the tangent planes common to the three ellipsoids

$$\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}+\frac{z^2}{a_3^2}=1,\ \frac{x^2}{a_2^2}+\frac{y^2}{a_3^2}+\frac{z^2}{a_1^2}=1,\ \frac{x^2}{a_3^2}+\frac{y^2}{a_1^2}+\frac{z^2}{a_2^2}=1\,,$$

and show that all of them are tangent planes to a sphere with centre at the origin.

HINT:

Find the condition for a plane $A_1x+A_2y+A_3z=1$ to be tangent to an ellipsoid. Show that the sphere in question is

$$x^2+y^2+z^2=\frac{a^2+b^2+c^2}{3}\,.$$

6.2.9 Show that the tangent planes which touch the conoid

$$\boldsymbol{r}=\{u\cos v,\ u\sin v,\ k[\tan v]^{\frac{1}{2}}\}$$

at points of a rectilinear generator, cut the x, y-plane in parallel straight lines. Is this result true for all conoids (see **6.2.3**)?

6.2.10 Consider the surface

$$\tilde{\boldsymbol{r}}=\{(v+p)\sin u+f(u),\ (v+q)\cos u+g(u),\ kv+h(u)\}\,,$$

where k, p and q are constants and $f(u)$, $g(u)$ and $h(u)$ are arbitrary functions of u. Find the curves on this surface for which the tangents are parallel to the rectilinear generators of the surface. In particular, determine the curves on this surface which have constant curvature.

HINT:

The rectilinear generators are defined by the curves $u=$const. The

direction of these generators is given by $\bar{r}_v$. If $r=r(s)$ is a required curve, then $r'(s)=\lambda\bar{r}_v$, where $\lambda=1/|\bar{r}_v|$. Hence,

$$r(s) = \left\{\lambda\int_0^s \sin u(t)\,dt\,,\quad \lambda\int_0^s \cos u(t)\,dt\,,\ k\lambda s\right\}, \qquad (*)$$

where $\lambda=1/[1+k^2]^{\frac{1}{2}}$, and u is a function of s.

Since $r'(s)=\lambda\bar{r}_v$, $\kappa^2=(u')^2/[1+k^2]$. Hence, if $\kappa=\text{const.}$,

$$u=\kappa s[1+k^2]^{\frac{1}{2}}+C\,,$$

where $C=$ const. Substitution of this value for u in $(*)$ yields the required curves.

6.2.11 If segments of constant length k are marked off along the normals to a surface S, the locus of their end-points $\bar{S}$ is a surface *parallel* to the given surface. The end-points on a given normal determine the corresponding points on these surfaces. Show that the tangent planes to these surfaces at corresponding points are parallel.

6.3 ENVELOPE AND EDGE OF REGRESSION

6.3.1 Find the envelopes of the following one-parameter families of surfaces where α is the family parameter:

(a) The planes $kx\cos\alpha+by\sin\alpha+z-c=0$;

(b) The spheres $x^2+y^2+(z+k\alpha)^2-\alpha^2=0$, $k\neq\pm1$;

(c) The paraboloids $r=\left\{u\cos v,\ u\sin v,\ \dfrac{u^2}{4\alpha}+\alpha\right\}$;

(d) The plane $r=\{u, v, \alpha^3u+\alpha^2v+1\}$;

where k, b and c are non-zero constants.

ANSWER:

(a) the elliptic cone $k^2x^2+b^2y^2-(z-c)^2=0$;

(b) for $k^2>1$, the cone $(k^2-1)(x^2+y^2)=z^2$, for $k^2<1$, no envelope ;

(c) the cone $x^2+y^2=z^2$;

(d) $27x^2(z-1)=4y^3$.

6.3.2 Determine the edge of regression for the envelope of the one parameter family of planes $z=\alpha^3x+\alpha^2y+\alpha$, where α is the parameter. What is the envelope?

ANSWER:

The intersection of the cylinders

$$3x-y^2=0 \quad \text{and} \quad 3yz+1=0\,.$$

6.3.3 A fixed point C on the z-axis is connected with a variable point P in the x, y-plane. Find the envelope of the planes which pass through P and are orthogonal to CP.

ANSWER:

The paraboloid of revolution $x^2+y^2=4cz$, where c denotes the distance between C and the origin.

6.3.4 Find the edge of regression of the envelope of the family of planes

$$x\sin\alpha-y\cos\alpha+z-k\alpha=0,$$

where α is the family parameter.

ANSWER:

The helix $\boldsymbol{r}=\{k\cos u,\ k\sin u,\ ku\}$.

6.3.5 Find

(a) the envelope,

(b) the edge of regression,

for the family of planes

$$3\alpha^2x-3\alpha y+z-\alpha^3=0\,.$$

ANSWER:

(a) $(2x^3-3xy+z)^2=4(x^2-y)^3$;

(b) $\boldsymbol{r}=\{u, u^2, u^3\}$.

6.4 DEVELOPABLE AND RULED SURFACES

6.4.1 Consider the curve $\boldsymbol{r}=\boldsymbol{r}(s)$.

(a) What surface is defined by the equation

$$\bar{\boldsymbol{r}}=\boldsymbol{r}(s)+(u-x)\boldsymbol{r}'(s)\,,$$

where u is a parameter?

(b) What are the coordinate curves of this surface?

HINT:

Show that the tangent planes are tangent to the rectilinear generators of this surface.

6.4.2 Consider the equation

$$\boldsymbol{r} = \{u \cos v,\ u \sin v,\ \sin 2v\}\ .$$

(a) Discuss the surface defined by this equation.
(b) Show that this surface is one-sided.

HINT:

(a) The ruled surface $z(x^2+y^2)=2xy$ is a cylindroid.
(b) Consider the curve

$$u = -2u_0 t + u_0\ , \qquad v = \pi t + v_0\ , \qquad 0 \leqslant t \leqslant 1\ ,$$

which lies on the given cylindroid and starts and finishes at the point $M_0(u_0, v_0)$. Determine the unit normal $\boldsymbol{N}_0$ to this surface at (u_0, v_0), and show that $\boldsymbol{N} = \boldsymbol{N}_0$ at $t=0$ and $\boldsymbol{N} = -\boldsymbol{N}_0$ at $t=1$.

6.4.3 Find the edge of regression for the developable surface which is the envelope of the tangential planes of the hyperbolic paraboloid $z=xy$ along the curve which is the section between the cylinder $x^2=y$ and the paraboloid.

HINT:

Rewrite the equations for the paraboloid and the curve of section in the forms

$$\boldsymbol{r} = \{u, v, uv\} \quad \text{and} \quad u^2 = v\ ,$$

respectively. Let u be the parameter of the curve, and show that required tangent planes are

$$u^2 x + uy - z - u^3 = 0\ ,$$

where u is the family parameter. Hence the edge of regression is $\bar{\boldsymbol{r}} = \{3u, -3u^2, -u^3\}$.

6.4.4 Consider the two parameter family of straight lines

$$x = tz + p, \quad y = pz + \frac{t^3}{3}, \tag{1}$$

where t and p are family parameters.

(a) Find the relationship between t and p such that the lines (1) are generators of a developable surface.

(b) Find the edge of regression of one of these developable surfaces.

(c) Find the locus of these edges of regression.

(d) What is the section of these developable surfaces with the x, y-plane.

HINT:

(a) Let $p=p(t)$ be the required function. Putting $z=u$, (1) can be written in the form

$$\boldsymbol{r} = \left\{p(t), \frac{t^3}{3}, 0\right\} + u\{t, p(t), 1\},$$

which is the equation of a ruled surface. Using the condition which ensures the developability of a ruled surface (see **5.4**), it is found that

$$p(t) = c \pm \frac{t^2}{2}, \quad c = \text{const.}$$

(b) The normal vector to the developable surface is

$$\boldsymbol{r}_t \times \boldsymbol{r}_u = \left\{t(t \pm u), (t \pm u), \left(\mp \frac{t^2}{2} - c - t^2\right)(t \pm u)\right\}.$$

The edge of regression is a singular curve of the surface and is defined by $\boldsymbol{r}_t \times \boldsymbol{r}_u = 0$, i.e., by $t = \pm u$. Hence the equation of the edge of regression is

$$\boldsymbol{r} = \left\{c \pm \frac{t^2}{2}, -\frac{t^3}{6} \pm ct, \pm t\right\}. \tag{2}$$

(c) Equation (2) defines a one-parameter family of curves with c as family parameter. Elimination of t and c gives

$$9(xz - y)^2 - 4z^6 = 0.$$

(d) $2(x-c)(x-4c)^2 \pm 9y^2 = 0$.

6.4.5 Show that the line which is defined by $y=ux-u^3$, $z=u^3y-u^6$, where u is a parameter, generates a developable surface.

HINT:
Write the equation of the surface in vector form.

6.4.6 Determine whether the surfaces
(a) $xyz=k^3$,
(b) $xy=(z-k)^2$,
are developable.

6.5 INVOLUTES AND EVOLUTES

6.5.1 Determine the involutes and evolutes of the circular helix

$$\boldsymbol{r}=\{k\cos t,\ k\sin t,\ bt\}\,.$$

ANSWER:
The involutes and evolutes are plane curves.

6.5.2 Find the curvature and torsion of the involutes of the curve $\boldsymbol{r}=\boldsymbol{r}(s)$.

HINT:
The equation of the involutes is

$$\boldsymbol{r}^*(s)=\boldsymbol{r}+(c-s)\boldsymbol{t}\qquad (c \text{ is a parameter})\,.$$

Differentiation of this equation twice with respect to s yields

$$\boldsymbol{t}^*=\boldsymbol{n},\ \kappa^*=(\kappa^2+\tau^2)^{\frac{1}{2}}/[\kappa(c-s)],\ \kappa^*\boldsymbol{n}^*=(-\kappa\boldsymbol{t}+\tau\boldsymbol{b})/[\kappa(c-s)]\,.$$

Hence,

$$\tau^*=(\kappa\tau'-\tau\kappa')/[(\kappa^2+\tau^2)\kappa(c-s)]\,.$$

6.5.3 If $\boldsymbol{r}=\boldsymbol{r}(s)$ is a plane curve, show that its evolutes are helices.

6.5.4 At a point P on the curve $\boldsymbol{r}=\boldsymbol{r}(s)$, a line $\boldsymbol{R}$ is drawn in the normal plane to make an angle φ with the principal normal at P.

(a) If $\boldsymbol{R}$ is tangent to a curve C_1 (an evolute of C), prove that $d\phi/ds = -\tau$, where s and τ are the arc length and torsion of C.

(b) Prove that for any evolute C_1 (as defined above) the ratio of curvature κ^* and torsion τ^* equals cot φ.

6.5.5 Consider the developable surface which is the envelope of the tangent planes of the sphere $x^2+y^2+z^2=k^2$ along the curve of section between a cylinder and the sphere.

(a) Find the edge of regression of this developable surface.

(b) Determine the section of this developable with the x, y-plane.

ANSWER:

(a) The evolute of the section between the sphere and the cylinder;

(b) The parabola $y^2=4k(k-x)$.

6.6 LENGTH AND THE FIRST FUNDAMENTAL FORM

6.6.1 On the surface $\boldsymbol{r}=\{k\cos v,\ k\sin v,\ u\}$, consider the curve $u=Kt, v=t$, where $K=$ const. Find the length of this curve between the points $t=t_1$ and $t=t_2$.

HINT:

On this surface $(ds)^2=k^2(dv)^2+(du)^2$, while on the curve $du=K\,dt$ and $dv=dt$. Hence, $s=[k^2+K^2]^{\frac{1}{2}}(t_2-t_1)$.

6.6.2 On the surface

$$\boldsymbol{r}=\{u\cos v,\ u\sin v,\ \tfrac{1}{2}u^2\},$$

consider the curve $v=ku$.

(a) Examine the form of this curve.

(b) Find the length of this curve from the origin to an arbitrary point $u=t$.

HINT:

(a) The curve lies on the surface of revolution generated by rotation of the parabola $2z=x^2$, $y=0$, about the z-axis. The projection of $v=ku$ on the x, y-plane is an Archimedean spiral.

(b) $s = \frac{\gamma t}{2}(1+\gamma^2 t^2)^{\frac{1}{2}} + \frac{1}{2}\log[\gamma t + (1+\gamma^2 t^2)^{\frac{1}{2}}]$, $\gamma = [1+k^2]^{\frac{1}{2}}$.

6.6.3 Consider the equation

$$\boldsymbol{r} = \{u\cos v,\ u\sin v,\ uv\}\,.$$

(a) What surface is represented by this equation?

(b) What are the curves of section between this surface and planes parallel to the x, y-plane?

(c) Find the length of the curve of section in the x, y-plane from the origin to an arbitrary point $u=t$.

ANSWER:

(a) The surface is the special cone where the rectilinear generators $v=$const. intersect the circular helices $u=$const. and at the same time cut the z-axis at a constant angle (see **6.1.6**).

(b) The hyperbolic spirals $r\theta=$const.

6.6.4 In the plane consider the affine coordinate system $\{O; \boldsymbol{e}_1, \boldsymbol{e}_2\}$, where O is the origin of the system and $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are vectors of length e_1 and e_2 such that $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = e_1 e_2 \cos\omega$.

(a) Find the expression for the first fundamental form of the plane with respect to this coordinate system.

(b) Find the formula for the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane.

(c) Find the expression for the length s of the curve $y=f(x)$ between the points x_0 and x_1.

ANSWER:

(a) $(\mathrm{d}s)^2 = (e_1)^2(\mathrm{d}x)^2 + 2e_1e_2\cos 2\omega \mathrm{d}x\mathrm{d}y + (e_2)^2(\mathrm{d}y)_2$, where x and y are the affine coordinates of any point in the plane, i.e., $\boldsymbol{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$.

(b) $d^2 = (e_1)^2(x_2-x_1)^2 + 2e_1e_2\cos\omega(x_2-x_1)(y_2-y_1) + (e_2)^2(y_2-y_1)^2$.

(c) $s = \int_{x_0}^{x_1} \{(e_1)^2 + 2e_1e_2\cos\omega f'(x) + (e_2)^2(f'(x))^2\}^{\frac{1}{2}}\mathrm{d}x$,

where $f'(x) = \mathrm{d}f(x)/\mathrm{d}x$.

6.6.5 Consider the equation

$$\boldsymbol{r} = \{2pu \cos v,\ 2qu \sin v,\ 2u^2(p \cos^2 v + q \sin^2 v)\},$$

where p and q are positive constants.

(a) What surface is represented by this equation?

(b) Determine the curves along which the tangent planes to the surface cut the x, y-plane at a constant angle.

(c) Determine the first fundamental form for this surface.

(d) Find the length of one of the curves $u = \text{const.}$

HINT:

(a) The paraboloid $qx^2 + py^2 = 2pqz$.

(b) Consider the normal to a tangent plane to show that for the required curves $u = \text{const.}$

(d) The curves $u = \text{const.}$ are closed.

6.6.6 Determine the first fundamental form for a plane with respect to the polar coordinates r and θ. Find the expression for the length of the curve $r = r(\theta)$ which lies in the plane.

ANSWER:

$$(\mathrm{d}s)^2 = (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2\,, \quad s = \int [r^2 + (r')^2]^{\frac{1}{2}}\,\mathrm{d}\theta, \quad r' = \mathrm{d}r/\mathrm{d}\theta\,.$$

6.7 ANGLES, NETS AND TRAJECTORIES

6.7.1 On the sphere

$$\boldsymbol{r} = \{k \cos u \cos v,\ k \cos u \sin v,\ k \sin v\}, \quad k > 0\,,$$

consider the curves $u = v$. Find the angles α and β between these curves and the curves $v = \text{const.}$ and $u = \text{const.}$, respectively. In particular, examine the cases for points of intersection on the equator and at the poles.

ANSWER:

$\cos \alpha = \cos v(1 + \cos^2 v)^{\frac{1}{2}}$ (see **5.7**). Since $g_{12} = 0$, $\beta = \frac{1}{2}\pi - \alpha$. The equator is defined by $v = 0$; hence, $\alpha = \beta = \frac{1}{4}\pi$. At the poles, $v = \pm\frac{1}{2}\pi$;

hence, $\alpha=\frac{1}{2}\pi$, $\beta=0$; the given curves are tangent to the lines of latitude at the poles.

6.7.2 Find the angle α between the curve $r=r(\theta)$ defined in **6.6.6** and the curves $\theta=\text{const}$.

HINT:

Show that $\tan\alpha=r/r'$.

6.7.3 Oblique coordinates are the affine coordinates of **6.6.4** with $e_1=e_2=1$. Find the angle α between the straight line $y=kx+a$ and the e_1-axis, where x and y are the components of a general point with respect to the oblique coordinates $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively.

ANSWER:

Using the results of **6.6.4**, one finds

$$\tan\alpha=\frac{k\sin\omega}{1+k\cos\omega}.$$

6.7.4 On the hyperbolic paraboloid $\boldsymbol{r}=\{u, v, uv\}$, consider the curves $u=\cos t$, $v=\sin t$ and $u=\tau$, $v=k\tau$, where t and τ are independent parameters.

(a) Find the points of intersection between the given curves, and determine the angle θ between them.

(b) For what values of k are the curves orthogonal?

ANSWER:

(a) The points of intersection are

$$(u, v)=(\pm[1+k^2]^{-\frac{1}{2}},\ \pm k[1+k^2]^{-\frac{1}{2}}).$$

The angle θ is given by

$$\cos\theta=\frac{1}{2\sqrt{2}}\sin 4t[\sin^4 t+\cos^4 t]^{-\frac{1}{2}},$$

where $\tan t=k$.

(b) $k=0, \pm 1, \infty$. These cases arise when the second curve is ob-

tained as the section of the paraboloid by the planes $y=0$, $y=\pm x$ and $x=0$, respectively.

6.7.5 On the surface with first fundamental form

$$(\mathrm{d}s)^2 = g_{11}(\mathrm{d}u^1)^2 + 2g_{12}\,\mathrm{d}u^1\,\mathrm{d}u^2 + g_{22}(\mathrm{d}u^2)^2\,,$$

consider the curves defined by the differential equation

$$P(u^1, u^2)(\mathrm{d}u^1)^2 + Q(u^1, u^2)\mathrm{d}u^1\,\mathrm{d}u^2 + R(u^1, u^2)(\mathrm{d}u^2)^2 = 0\,. \qquad (1)$$

Find the conditions under which the curves (1) form an orthogonal net (see **5.7**).

HINT:

The condition of orthogonality for two curves $u^i = u^i(t)$ and $v^i = v^i(t)$ $(i=1, 2)$ on the surface is (see **5.7**)

$$g_{ik}\,\mathrm{d}u^i\,\mathrm{d}v^k = 0\,.$$

The directions of the tangents $\mathrm{d}u^1/\mathrm{d}u^2$ and $\mathrm{d}v^1/\mathrm{d}v^2$ to the curves defined by (1) are the roots of (1); thus,

$$\frac{\mathrm{d}u^1}{\mathrm{d}u^2} + \frac{\mathrm{d}v^1}{\mathrm{d}v^2} = -\frac{Q}{P}\,, \qquad \frac{\mathrm{d}u^1}{\mathrm{d}u^2}\,\frac{\mathrm{d}v^1}{\mathrm{d}v^2} = \frac{R}{P}\,.$$

Hence, the required condition is

$$g_{11}R - g_{12}Q + g_{22}P = 0\,.$$

6.7.6 For the surface $\boldsymbol{r} = \boldsymbol{r}(u^1, u^2)$, find the orthogonal trajectories of its coordinate curves.

ANSWER:

For the curves $u^1 = \text{const.}$ and $u^2 = \text{const.}$, the orthogonal trajectories are defined by

$$g_{12}\mathrm{d}u^1 + g_{22}\mathrm{d}u^2 = 0\,, \qquad g_{11}\mathrm{d}u^1 + g_{12}\mathrm{d}u^2 = 0\,,$$

respectively.

6.7.7 Find the orthogonal trajectories of the rectilinear generators of the tangent surface of a circular helix.

ANSWER:

The trajectories are the involutes of the given circular helix.

6.7.8 Find the orthogonal trajectories of the coordinate curves of the surface $\boldsymbol{r}=\{u\cos v,\ u\sin v,\ u+v\}$.

6.7.9 Find the differential equation which defines an orthogonal net on a surface with first fundamental form $(\mathrm{d}s)^2=g_{11}(\mathrm{d}u^1)^2+g_{22}(\mathrm{d}u^2)^2$.

6.7.10 Find the orthogonal trajectories of

(a) the coordinate curves of the surface

$$\boldsymbol{r}=\{v\cos u-k\sin u,\ v\sin u+k\cos u,\ ku\}\ ;$$

(b) the rectilinear generators of the ruled surface

$$\bar{\boldsymbol{r}}=\boldsymbol{r}(s)+u\boldsymbol{e}(s)\,.$$

ANSWER:

(a) $v=ku+C$ and $v=k\sqrt{2}\tan(u\sqrt{2}+C')$, where C and C' are constants.

(b) $u+\int\cos\theta\,\mathrm{d}s=\text{const.}$, where $\cos\theta=\boldsymbol{e}\cdot\boldsymbol{r}'(s)$.

6.7.11 Show that on the helicoid $\boldsymbol{r}=\{u\cos v,\ u\sin v,\ kv\}$, the solution curves of the differential equation $(\mathrm{d}u)^2-(u^2+k^2)(\mathrm{d}v)^2=0$ form an orthogonal net.

6.7.12 On a surface with first fundamental form $(\mathrm{d}s)^2=g_{ik}\mathrm{d}u^i\mathrm{d}u^k$, consider the curves $P\mathrm{d}u^1+Q\mathrm{d}u^2=0$, where P and Q are given functions of u^1 and u^2.

(a) Find the differential equation defining the orthogonal trajectories of the given curves.

(b) Determine the orthogonal trajectories of the curves $u+v=$ const. lying on the unit sphere

$$\boldsymbol{r}=\{\cos u\cos v,\ \cos u\sin v,\ \sin u\}\quad(\text{see }\mathbf{6.1.3})\,.$$

ANSWER:

(a) $(Pg_{12}-Qg_{11})\mathrm{d}u^1+(Pg_{22}-Qg_{12})\mathrm{d}u^2=0$.

(b) Since $g_{11}=\cos^2 v$, $g_{12}=0$, $g_{22}=1$ and $P=Q=1$, the required orthogonal trajectories are

$$u+C=\tan v, \quad C=\text{const.}$$

6.7.13 Do the two families of curves

$$\sin u+p(v+1)=0 \quad \text{and} \quad \frac{p}{3\sin^2 u}+v=q \quad (p, q=\text{const.})$$

form an orthogonal net on the surface

$$\boldsymbol{r}=\left\{\sin u\cos v,\ \sin u\sin v,\ \cos u+\log\tan\frac{u}{2}+1\right\}?$$

6.7.14 Find the differential equation for the isogonal trajectories (see **5.7**) of the coordinate curves of the surface with first fundamental form $(\mathrm{d}s)^2=g_{ik}\mathrm{d}u^i\mathrm{d}u^k$.

ANSWER:

For the curves $u^2=\text{const.}$, the required equation is

$$g_{1i}\mathrm{d}u^i=(g_{11})^{\frac{1}{2}}[g_{ik}\mathrm{d}u^i\mathrm{d}u^k]^{\frac{1}{2}}\cos\theta,$$

where θ is the angle between the trajectories and the curves $u^2=\text{const.}$ A similar result holds for the curves $u^1=\text{const.}$

6.7.15 Find the isogonal trajectories of the lines of longitude on a surface of revolution.

ANSWER:

Let the surface of revolution (see **6.2.5**) have the form $\boldsymbol{r}=\{u\cos v, u\sin v, f(u)\}$. Using **6.7.14**, the required trajectories are found to be

$$v\cot\theta+C=\pm\int[1+(f')^2]^{\frac{1}{2}}\frac{\mathrm{d}u}{u} \quad (C=\text{const.}),$$

where $f'=\mathrm{d}f(u)/\mathrm{d}u$ and θ is defined in **6.7.14**.

6.7.16 The *torus* is the surface of revolution with the equation

$f(u)=[r^2-(R-u)^2]^{\frac{1}{2}}$, where $r<R$ are constants (see **6.7.15**). Find the isogonal trajectories of the lines of longitude of the torus.

ANSWER:

$u=p[1+q\cos m(v-v_0)]^{-1}$, $v_0=$ const., where $p=(R^2-r^2)R^{-1}$, $q=rR^{-1}$ and $m=[R^2-r^2]^{\frac{1}{2}}r^{-1}\cot\theta$, and θ is defined in **6.7.14**.

6.7.17 Find the curves which bisect the angles between the coordinat curves of a surface with first fundamental form $(\mathrm{d}s)^2=g_{ik}\mathrm{d}u^i\mathrm{d}u^k$.

ANSWER:

The two families are defined by

$$[g_{11}]^{\frac{1}{2}}\mathrm{d}u^1+[g_{22}]^{\frac{1}{2}}\mathrm{d}u^2=0,\quad [g_{11}]^{\frac{1}{2}}\mathrm{d}u^1-[g_{22}]^{\frac{1}{2}}\mathrm{d}u^2=0\,.$$

6.7.18 Find the curves which bisect the angles between the coordinate curves of the following surfaces:

(a) $\boldsymbol{r}=\{k\cos u\sin v,\ k\sin u\sin v,\ k\cos v\}$;

(b) $\boldsymbol{r}=\{u\cos v,\ u\sin v,\ kv\}$;

(c) $\boldsymbol{r}=\{u\cos v,\ u\sin v,\ u+v\}$.

ANSWER:

(a) $u\pm\log\tan\dfrac{v}{2}=$ const. ;

(b) $\log\{u+[u^2+k^2]^{\frac{1}{2}}\}\pm v=$ const. ;

(c) $\sqrt{2}\log\{u+[u^2+1]^{\frac{1}{2}}\}\pm v=$ const.

6.7.19 Why is an isometric (isothermic) net an orthogonal net?

HINT:

See **5.7**.

6.7.20 Consider the equation

$$\boldsymbol{r}=\left\{\pm\left[\frac{k-b}{k}uv\right]^{\frac{1}{2}},\ \pm\frac{1}{b}\left[\frac{b-k}{b}(1+ku)(1+kv)\right]^{\frac{1}{2}},\ \frac{b-k}{2bk}(1+ku+kv)\right\},$$

where $k > b > 0$ are constants.

(a) What surface is represented by this equation?

(b) Show that the coordinate curves form an isometric net.

HINT:

(a) The surface is a paraboloid.

(b) $(ds)^2 = \frac{k-b}{4b^2}(u-v)[f(u)(du)^2 - f(v)(dv)^2]$,

where $f(u) = \frac{k(k-b)u-b}{u(1+ku)}$. From the condition that the expressions under the square root sign in $\boldsymbol{r}$ must be positive, it is seen that the parameters u and v vary in the regions G_1 and G_2 defined by

$$v < -\frac{1}{k}, \ -\frac{1}{k} < u < 0 \text{ and } u < -\frac{1}{k}, \ -\frac{1}{k} < v < 0,$$

respectively. In G_1, introduce the new coordinates $\bar{u}$ and $\bar{v}$ defined by

$$\bar{u} = \int [f(u)]^{\frac{1}{2}} du, \quad \bar{v} = \int [-f(v)]^{\frac{1}{2}} dv.$$

Then (1) takes the form

$$(ds)^2 = \lambda(\bar{u}, \bar{v})[(d\bar{u})^2 + (d\bar{v})^2].$$

A similar argument applies for G_2.

6.7.21 Show that, if the coordinate curves of a surface with first fundamental form $(ds)^2 = g_{ik}du^1du^2$ generate an isometric net, then the curves $u^1 + u^2 = \text{const.}$ and $u^1 - u^2 = \text{const.}$ bisect the angles of this net and form an isometric net themselves.

6.7.22 Find the surface area of the sphere

$$\boldsymbol{r} = \{k \cos u \cos v, \ k \cos u \sin v, \ k \sin u\}.$$

HINT:

See NOTE in **5.7**.

6.8 THE SECOND FUNDAMENTAL FORM AND CURVATURE

6.8.1 Determine the second fundamental form for the following surfaces

(a) $\boldsymbol{r}=\{k\cos u\cos v,\ k\cos u\sin v,\ k\sin u\}$ (the sphere) ;
(b) $\boldsymbol{r}=\{k\sin u\cos v,\ b\sin u\sin v,\ c\cos u\}$ (the ellipsoid) ;
(c) $\boldsymbol{r}=\{f(u),\ g(u),\ v\}$ (the general cylinder) ;
(d) $\boldsymbol{r}=\{u,\ v,\ kuv\}$ (the hyperbolic paraboloid) ;
(e) $\boldsymbol{r}=\{u\cos v,\ u\sin v,\ f(v)\}$ (the conoid) ;
(f) $\boldsymbol{r}=\{u\cos v,\ u\sin v,\ f(u)\}$ (the surface of revolution) ;
(g) The tangent surface of the curve (Viviani's curve)

$$\boldsymbol{r}=\left\{k(1+\cos u),\ k\sin u,\ 2k\sin\frac{u}{2}\right\};$$

(h) The surface generated by the principal normals of a circular helix.

6.8.2 Repeat **6.8.1** for the surfaces
(a) $z(x^2+y^2)=2xy$ (the cylindroid) ;
(b) $x^2+y^2=(\tan^2\alpha)z^2$ (the cone with z-axis as axis) ;
(c) $27x^2(z-1)=4y^3$;
(d) $z=k_1\ \dfrac{y}{x}\ +[x^2+y^2]^{\frac{1}{2}}\cot\alpha$ (see **6.1.6**) ;
(e) $xyz=k^3$.

6.8.3 Show that the relationship (5.8.1) holds for the following curves on the following surfaces:
(a) the curve $\phi=\theta$ on the unit sphere with longitude ϕ and colatitude θ.
(b) the spherical indicatrix for the tangents to the curve $\boldsymbol{r}=\{2t-\sin 2t,\ -\cos 2t,\ 4\sin t\}$ on the unit sphere (see **4.34**).
(c) the curve $u=v$ on the surface

$$\boldsymbol{r}=\left\{\sin u\cos v,\ \sin u\sin v,\ \cos u+\log\tan\frac{u}{2}\right\}.$$

6.9 NORMAL CURVATURE

6.9.1 Find the curvature κ at the origin of the curve of section between the surface

$$z=\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2},$$

where α and β are constants, and the planes $z=ky$.

ANSWER:

The normal section of the surface which has common tangent with the given curve at the origin is the parabola $z=x^2/\alpha^2$, $y=0$. Using Meusnier's Theorem (see **5.9**), it follows that

$$\kappa = \frac{2(1+k^2)^{\frac{1}{2}}}{\alpha^2 k^2}, \quad k = \tan\left(\frac{\pi}{2} - \theta\right),$$

where θ is the angle between the normal section and the plane of the given curve at the origin.

6.9.2 Let the curve C be the intersection between two surfaces S_1 and S_2. At the point P of C, let κ_1 and κ_2 be the normal curvatures of S_1 and S_2 in the direction of the tangent to C at P. If θ is the angle between the normals to the surface at P, determine the curvature κ of C at P.

ANSWER:

$\kappa^2 = (\kappa_1^2+\kappa_2^2-2\kappa_1\kappa_2 \cos\theta)/\sin\theta$.

6.9.3 Determine the normal curvature at points on the following curves on the following surfaces:

(a) The curve $\phi = \theta$ on the unit sphere with longitude ϕ and colatitude θ.

(b) The curve $u = v$ on the tangent surface of the curve (Viviani's curve)

$$\boldsymbol{r} = \left\{k(1+\cos u),\ k \sin u,\ 2k \sin\frac{u}{2}\right\}.$$

(c) The curve $u = v^2$ on the surface $\boldsymbol{r} = \{u, v, kuv\}$.

6.10 PRINCIPAL CURVATURES AND DIRECTIONS. UMBILICAL POINTS

6.10.1 Show that the principal directions on a surface form an orthogonal net.

HINT:

Use the condition for the orthogonality of two curves.

6.10.2 Find the principal curvatures of the paraboloid $xy=kz$ at the points of intersection of the paraboloid with the hyperboloid $x^2+y^2-z^2+k^2=0$.

ANSWER:

$$\kappa_1, \kappa_2 = \frac{k}{z^2}(1\pm\sqrt{2}).$$

6.10.3 Determine the ratio of the principal curvatures of the paraboloid.

ANSWER:

$$\frac{\kappa_1}{\kappa_2} = -2.$$

6.10.4 Show that, if $g_{12}=0$ and $h_{12}=0$, the principal curvatures of a surface are defined by

$$\kappa_1 = \frac{h_{11}}{g_{11}}, \qquad \kappa_2 = \frac{h_{22}}{g_{22}}.$$

6.10.5 Show that the sum of the normal curvatures for two perpendicular directions at a point on a given surface is constant.

HINT:

Use Euler's formula to show that the constant value is $\kappa_1+\kappa_2$, the sum of the principal curvatures.

6.10.6 Determine the normal curvature of the paraboloid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=z$ at the origin.

ANSWER:

The normal sections of the paraboloid at the origin are the sections of the paraboloid by the planes $y=kx$, $k=$const. The principal directions at the origin are the directions of the x- and y-axes. Let ϕ be the angle between the normal section and the x-axis. Using Euler's

formula, we obtain

$$\kappa_n = \frac{2}{\alpha^2\beta^2}\,\frac{\beta^2+\alpha^2k^2}{1+k^2}\,, \text{ where } k=\tan\phi\,.$$

6.10.7 Find necessary and sufficient conditions for a point on the surface $z=f(x, y)$ to be an umbilical point.

ANSWER:

$$\frac{r}{1+p^2} = \frac{s}{qp} = \frac{t}{1+q^2}\,,$$

where p, q, r, s and t are the Monge abbreviations for the derivatives of z.

6.10.8 Determine the umbilical points on the paraboloid $2z = \frac{x^2}{\alpha} + \frac{y^2}{\beta}$.

ANSWER:

The points are $\left(0, \pm[\beta(\alpha-\beta)]^{\frac{1}{2}}, \frac{\alpha-\beta}{2}\right)$ if $\alpha>\beta$, and

$\left(\pm[\alpha(\beta-\alpha)]^{\frac{1}{2}}, 0, \frac{\beta-\alpha}{2}\right)$ if $\beta>\alpha$.

6.10.9 Find the umbilical points on the ellipsoid

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} = 1\,, \quad \alpha>\beta>\gamma>0\,.$$

6.10.10 Show that all points on the curve $y^2=2kx, z=0$, are umbilical points on the surface

$$kz^2+(2x+k)(y^2-akx)=0\,.$$

6.11 GAUSSIAN AND AVERAGE CURVATURE

6.11.1 Let S be the surface of revolution of a plane curve about the z-axis. Find the Gaussian curvature K of this surface.

ANSWER:

Take the equation of S in the form

$$\boldsymbol{r} = \{\rho(s)\cos v,\ \rho(s)\sin v,\ f(s)\}\ ,$$

where s is the arc length of the lines of longitude. Since, on a line of longitude, $(\mathrm{d}s)^2 = \{(\rho')^2 + (f')^2\}\,\mathrm{d}s^2$, it is seen that $\rho'\rho'' + f'f'' = 0$. Using this relationship and the formula for K (see **5.11**), it is found that

$$K = -\rho''/\rho\ .$$

6.11.2 Determine the Gaussian curvature for the following surfaces:

(a) the paraboloid $\boldsymbol{r} = \{u\cos v,\ u\sin v,\ ku^2\}$;

(b) the torus (see **6.7.16**) ;

(c) the surface generated by rotating the curve $z = k\sqrt{x}$, $y = 0$, about the z-axis

(d) the surface (the pseudosphere) generated by rotating the tractrix

$$\boldsymbol{r} = \left\{k\sin v,\ 0,\ k\left(\operatorname{logtan}\frac{v}{2} + \cos v\right)\right\}$$

about the z-axis.

ANSWER:

(a) On a line of longitude $(\mathrm{d}s)^2 = (1+4k^2u^2)(\mathrm{d}u)^2$. Hence, $K = 4k^2/(1+4k^2u^2)^2$;

(b) $K = \cos(s/r)\,[r\{R + r\cos(s/r)\}]^{-1}$;

(c) $K = -2k^2[x(4x+k^2)^2]^{-1}$;

(d) $K = -k^{-2}$.

6.11.3 Find the Gaussian and average curvatures for the following surfaces:

(a) $\boldsymbol{r} = \{k(u+v),\ b(u-v),\ uv\}$;

(b) Scherk's surface $z = 1/k\{\log(\cos kx) - \log(\cos ky)\}$;

(c) the binormal surface of the curve $\boldsymbol{r} = \boldsymbol{r}(s)$.

ANSWER:

(a) $K = -4k^2b^2/g^2$, $H = 4kb(k^2 - b^2 + uv)/g^{\frac{3}{2}}$, where

$$g = 4k^2b^2 + k^2(u-v)^2 + b^2(u+v)^2\ ;$$

(b) $K=-\{k\sec kx\sec ky/(1+\tan^2 kx+\tan^2 ky)\}^2$, $H=0$;
(c) $K=-\tau/(1+\tau^2u^2)^2$, $H=\{\kappa+\kappa\tau^2u^2-\tau'u\}/(1+\tau^2u^2)^{\frac{3}{2}}$, where u is a parameter.

6.11.4 Find the Gaussian curvature for the surface

$$e^z=\sin\{(x^2+y^2)^{\frac{1}{2}}\}\,.$$

ANSWER:

$$K=-\frac{\sin 2u}{2u},\quad u^2=x^2+y^2\,.$$

6.11.5 Find the average curvature of the tangent surface to the curve $\boldsymbol{r}=\boldsymbol{r}(s)$.

ANSWER:
$H=\tau/(u\kappa)$, where u is a parameter.

6.11.6 Show that on a right conoid the Gaussian curvature satisfies the inequality $-1<K\leqslant 0$.

HINT:
See **6.2.3**.

6.11.7 Find the surface of revolution with zero average curvature.

HINT:
Use the general equation defining a surface of revolution and the fact that $H=\frac{1}{2}(\kappa_1+\kappa_2)$ (see **5.11**).

ANSWER:
The surface is a catenoid: the surface of revolution generated by rotating the catenary

$$x=c\cosh\left(\frac{z-k}{c}\right),\quad y=0\,,$$

about the z-axis.

6.11.8 Find surfaces of revolution with constant positive Gaussian curvature $1/k^2$ and show that the sphere is one of these surfaces.

HINT:

Determine the generating curve of the surfaces in the x, z-plane and use the fact that $K=\kappa_1\kappa_2$ (see **5.11**). Recall that one of the principal curvatures equals the radius of curvature of the generating curve and use the Theorem of Meusnier to determine the other.

6.11.9 Find surfaces of revolution with constant negative Gaussian curvature $-1/k^2$ and show that the pseudosphere is one of these surfaces.

6.11.10 Consider the family of surfaces generated by translating a surface of revolution along its axis of revolution.

(a) Determine the surfaces of revolution which cut each of the given surfaces perpendicularly and which have their axes common with the given axis.

(b) Show that the Gaussian curvatures K and K^* of the given and required surfaces satisfy the condition $K=-K^*$.

ANSWER:

(a) Let the z-axis be the axis of revolution and the given family of surfaces be

$$\boldsymbol{r}=\{u\cos v\,,\ u\sin v\,,\ f(u)+h\}\,,$$

where h is the family parameter. Let the required surface be

$$\boldsymbol{r}=\{u\cos v\,,\ u\sin v\,,\ \phi(u)\}\,.$$

The surfaces will be orthogonal if the lines of longitude are orthogonal, i.e., if

$$\phi'(u)f'(u)+1=0\,. \qquad (*)$$

Hence, $\phi(u)=-\displaystyle\int\frac{du}{f'(u)}$.

(b) Use **6.11.1** and equation (*).

6.11.11 Consider the parallel surfaces S and $\bar{S}$ for which the distance between corresponding points is k. Let K, H and $\bar{K}$, $\bar{H}$ denote the Gaussian and average curvatures of S and $\bar{S}$, respectively.

(a) Find $\bar{H}$ and $\bar{K}$ in terms of H and K.

(b) Show that $\bar{S}$ is a developable surface if and only if S is a developable surface.

HINT:

(a) First show that

$$\bar{g}_{ik} = (1-k^2K)g_{ik} + k(kH-2)h_{ik}\,, \qquad (*)$$

and

$$\bar{h}_{ik} = kKg_{ik} + k(1-kH)h_{ik}\,.$$

Using these results, it follows that

$$\bar{K} = K/(1-kH+k^2K)\,,$$

$$\bar{H} = (H-2kK)/(1-kH+k^2K)\,.$$

6.12 THE THEOREM OF GAUSS. CHRISTOFFEL SYMBOLS

6.12.1 Show that

$$g_{ij}g^{ik} = \delta_j^k \qquad (i, j, k = 1, 2)$$

where δ_j^k is the Kronecker delta symbol:

$$\delta_j^k = \begin{cases} 1 & (j=k)\,, \\ 0 & (j\neq k)\,. \end{cases}$$

6.12.2 Show that (Theorem of Gauss)

$$g^{\frac{1}{2}}K = \frac{\partial}{\partial u^1}\left(\frac{g^{\frac{1}{2}}}{g_{22}}\Gamma^1_{22}\right) - \frac{\partial}{\partial u^2}\left(\frac{g^{\frac{1}{2}}}{g_{22}}\Gamma^1_{12}\right)\,,$$

and verify that

$$\frac{\partial}{\partial u^1}(\log g) = \Gamma^1_{11} + \Gamma^2_{12}\,, \quad \frac{\partial}{\partial u^2}(\log g) = \Gamma^1_{12} + \Gamma^2_{22}\,.$$

6.12.3 Show that for the surface $z = f(x, y)$,

$$\Gamma^1_{11} = pr/\phi, \quad \Gamma^1_{12} = ps/\phi, \quad \Gamma^1_{22} = pt/\phi, \quad \Gamma^2_{11} = qr/\phi,$$
$$\Gamma^2_{12} = qs/\phi, \quad \Gamma^2_{22} = qt/\phi, \quad K = (rt - s^2)/\phi^2 ,$$

where $\phi = 1 + p^2 + q^2$ and p, q, r, s and t are the abbreviations of Monge for the first and second derivatives of $f(x, y)$ (see **5.6** and **5.8**).

6.12.4 Find the Gaussian curvature for the surfaces with first fundamental forms:

(a) $(ds)^2 = (du)^2 + G(u)(dv)^2$;

(b) $(ds)^2 = \lambda(u, v)\{(du)^2 + (dv)^2\}$;

(c) $(ds)^2 = 2F\,du\,dv$.

ANSWER:

By the Theorem of Gauss, K is given by:

(a) $\{G(u)\}^{-\frac{1}{2}} \dfrac{d^2 G^{\frac{1}{2}}}{du^2}$;

(b) $-\dfrac{1}{2\lambda}\left\{\dfrac{\partial^2 \log \lambda}{\partial u^2} + \dfrac{\partial^2 \log \lambda}{\partial v^2}\right\}$;

(c) $-\dfrac{1}{F}\dfrac{\partial^2 \log F}{\partial u\,\partial v}$.

6.12.5 Using the Riemann symbols

$$R_{ijkl} = R^m_{ijk} g_{ml} \quad \text{(with summation over } m\text{)},$$

where

$$R^m_{ijk} = \frac{\partial}{\partial u^j}(\Gamma^m_{ik}) - \frac{\partial}{\partial u^i}(\Gamma^m_{jk}) + \Gamma^n_{ik}\Gamma^m_{nj} - \Gamma^n_{jk}\Gamma^m_{ni}$$

(with summation over n) show that the Gaussian curvature satisfies

$$K = \frac{R_{1212}}{g} .$$

6.13 CONJUGATE DIRECTIONS AND ASYMPTOTIC LINES

6.13.1 Consider the developable surface which is tangent to the paraboloid $\boldsymbol{r} = \{u, v, uv\}$ along the curve $u^i = u^i(t)$ $(i = 1, 2)$. Determine the direction of the generators of the developable surface.

ANSWER:

From **5.13** it is seen that

$$du^1 dv^2 + du^2 dv^1 = 0$$

where du^1/du^2 and dv^1/dv^2 are the directions of the tangents to the given curve at a point P and of the generator of the developable surface which passes through P, respectively. Hence, the direction of the generator is

$$\frac{dv^1}{dv^2} = -\frac{\dot{u}^1(t)}{\dot{u}^2(t)}.$$

6.13.2 On the surface

$$\boldsymbol{r} = \{\sin u \cos v,\ \sin u \sin v,\ \cos u + \operatorname{logtan} \tfrac{1}{2}u\},$$

consider the family of curves $u + \log(\sec v + \tan v) = \text{const}$. Find the family of curves which is conjugate with respect to the given curves.

ANSWER:

Using the usual formula, it is seen that

$$\frac{du}{dv}\frac{d\bar{u}}{d\bar{v}} = \sin^2 u$$

where du/dv and $d\bar{u}/d\bar{v}$ are the directions of the tangents to the given curves and the conjugate curves, respectively. Hence, for the required curves, $\cot u = \sin v + \text{const}$.

6.13.3 Determine the asymptotic lines on the following surfaces:

(a) $\boldsymbol{r} = \{u \cos v,\ u \sin v,\ k \log u\}$;

(b) $z = (x^2 + y^2)^{-1}$;

(c) $\boldsymbol{r} = \{u \cos v,\ u \sin v,\ k \cos nv\}$, $n = \pm 1, \pm 2, \ldots$;

(d) $\boldsymbol{r} = \left\{(1+\cos u)\cot v,\ 1+\cos u,\ \frac{\sin u}{\sin v}\right\}$;

(e) $\boldsymbol{r} = \{(1+u)\cosh v,\ (1-u)\sinh v, u\}$;

(f) $\boldsymbol{r} = \{(1+u)\cos v,\ (1-u)\sin v, u\}$;

(g) $\boldsymbol{r} = \{v \cos u - \sin u,\ v \sin u + \cos u, u\}$;

(h) $\boldsymbol{r} = \left\{\frac{v}{\cosh u}, \frac{uv}{\cosh u}, \arctan v\right\}$;

(i) $\boldsymbol{r} = \{3u+3v, 3u^2+3v^2, 2u^3+2v^3\}$.

ANSWER:

In the following solutions C_1 and C_2 denote constants.

(a) $u=C_1 e^{-v}$, $u=C_2 e^{v}$ (the projections of the asymptotic lines on the x, y-plane are logarithmic spirals);

(b) Putting $x = u\cos v$ and $y = u \sin v$, the asymptotic lines are found to be $u=C_1 \exp(-v/\sqrt{3})$, $u=C_2 \exp(v/\sqrt{3})$, where $\exp z = e^z$;

(c) $v=C_1$, $u=C_2\sqrt{\sin nv}$;

(d) $\cot \frac{1}{2}u = C_1 \sin^2 \frac{1}{2}v$, $\cot \frac{1}{2}u = C_2 \cos^2 \frac{1}{2}v$;

(e) $v = C_1$, $u = \dfrac{C_2 \cosh v - \sinh v}{C_2 \cosh v + \sinh v}$;

(f) $v = C_1$, $u = \dfrac{C_2 \cos v - \sin v}{C_2 \cos v + \sin v}$;

(g) $u=C_1$, $u-2v=C_2$;

(h) $v=\sinh(C_1+u/\sqrt{2})$, $v=\sinh(C_2-u/\sqrt{2})$;

(i) $u+v=C_1$, $u-v=C_2$.

6.13.4 Find the asymptotic lines on the conoid.

HINT:

See **6.2.3**.

ANSWER:

$y = C_1 x$, $x^2 = C_2 f'(y/x)$, $C_1, C_2 = \text{const}$. The dash denotes differentiation with respect to the argument of f.

6.13.5 Determine the asymptotic lines on the following surfaces:

(a) $\boldsymbol{r} = \{u\cos v, \sin v, kv\}$;

(b) $\boldsymbol{r} = \{k\cdot(u+v), b\cdot(u-v), uv\}$;

(c) $\boldsymbol{r} = \{(u+v)(3+2u^2+2v^2-8uv),$
$(v-u)(3+2u^2+2v^2+8uv), 12uv\}$.

ANSWERS:

For these three surfaces the coordinate curves are the asymptotic lines.

6.13.6 Find the differential equation defining the asymptotic lines on a surface S given in cylindrical coordinates.

SOLUTION: The equation of S has the form

$$\boldsymbol{r} = \{r\cos\theta,\ r\sin\theta,\ z(r,\theta)\}\,,$$

where r, θ, and z are cylindrical coordinates.

The required differential equation is

$$\frac{\partial^2 z}{\partial r^2}(\mathrm{d}r)^2 + 2\left(\frac{\partial^2 z}{\partial r\,\partial\theta} - \frac{1}{r}\frac{\partial z}{\partial\theta}\right)\mathrm{d}r\,\mathrm{d}\theta + \left(r\frac{\partial z}{\partial r} + \frac{\partial^2 z}{\partial\theta^2}\right)(\mathrm{d}\theta)^2 = 0\,.$$

6.13.7 Determine the asymptotic lines for the surfaces:

(a) $z = mxy$, $m =$ constant;

(b) $zx^2 = ky^2$;

(c) $z = y\sin x$;

(d) $z = x^m y^n$, m and n constants;

(e) $z = \arctan(y/x) - \frac{1}{2}\log(x^2+y^2)$.

ANSWER:

(a) $x = C_1$, $y = C_2$;

(b) $y = C_1 x$, $y = C_2 x^3$;

(c) $x = C_1$, $y^2\cos x = C_2$;

(d) $y = C_1 x^\alpha$, $y = C_2 x^\beta$, where α and β are the roots of the quadratic equation $n(n-1)t^2 + 2mnt + m(m-1) = 0$;

6.13.8 Consider the curve $\boldsymbol{r} = \{t^n, t^{n-1}, t^{n-2}\}$, where n is an integer. Find the asymptotic lines of the surface which is defined as the locus of the midpoints of all the chords of the given curve.

ANSWER:

$$u^{\frac{n-1}{2}} \pm v^{\frac{n-1}{2}} \qquad x = \text{const.}$$

6.13.9 Consider the straight line

$$z = mx, \quad y = nx\,. \tag{*}$$

(a) Find the locus of the straight lines which are parallel to the y,z-plane and cut the straight line (*).

(b) Determine the asymptotic lines on the surface found in (a).

ANSWER:

(a) The general form of such a surface is

$$z = y\phi(x) + x\{m - n\phi(x)\},$$

where $\phi(x)$ is an arbitrary function of x.

(b) $x = C_1,\ y = nx + C_2[\phi'(x)]^{-\frac{1}{2}}$.

6.13.10 Find the function $f(x)$ such that the asymptotic lines on the surface $z = f(x) - f(y)$ form an orthogonal net.

ANSWER:

The asymptotic lines are defined by the equation

$$\frac{dy}{dx} = \pm(f''(x)[f''(y)]^{-1})^{\frac{1}{2}}.$$

Hence, the orthogonality of the asymptotic lines implies (see **6.7.5**)

$$f''(x)[1 + (f'(y))^2] = f''(y)[1 + (f'(x))^2],$$

from which it follows that the equation of the surface is

$$kz = \log\left\{\frac{\cos(ky + b)}{\cos(kx + b)}\right\}.$$

6.13.11 Find the surface $z = F(x, y)$ with asymptotic lines defined by $y = C_1$, $xf(y) = C_2$, where $f(y)$ is an arbitrary function of y.

ANSWER:

The given curves are defined by the differential equation

$$f(y)dydx + xf'(y)(dy)^2 = 0.$$

Comparing this equation with the equation for the asymptotic lines on the surface $z = F(x, y)$, it follows that

$$\frac{\partial^2 F}{\partial x^2} = 0\,, \quad 2xf'(y)\frac{\partial^2 F}{\partial x\,\partial y} = f(y)\frac{\partial^2 F}{\partial y^2}\,.$$

Hence, $z=F(x, y)=kx+by+c+px\int[f(y)]^2\,dy$, where k, b, c and p are constants.

6.13.12 Consider the surface

$$z = f(x)y^2+ky+g(x)\,,$$

where $f(x)$ and $g(x)$ are given functions of x. Show that the asymptotic lines on this surface are defined by the differential equation

$$2f\left(\frac{dy}{dx}\right)^2 + f''y^2+4f'y\,\frac{dy}{dx} + g'' = 0\,.$$

{See: R. Godeau and D. S. Mitrinovitch, "Sur certaines surfaces dont les lignes asymptotiques se determinent par quadratures", *Mathesis* **51** (1937), 115–116}.

6.13.13 Find the torsion of the asymptotic lines on the surface $2z=kx^2+2hxy+by^2$.

HINT:

Using the Beltrami–Enneper Theorem (see **5.13**) it is found that

$$\tau = \pm(h^2-kb)[1+(kx+hy)^2+(hx+by)^2]^{-1}\,.$$

6.13.14 Determine the torsion of the asymptotic lines on the surfaces:

(a) $\boldsymbol{r} = \{u\cos v,\ u\sin v,\ kv\}$;

(b) $\boldsymbol{r} = \{u\cos v,\ u\sin v,\ u^{-1}\}$.

ANSWER:

(a) $\tau = \pm k(u^2+k^2)^{-1}$;

(b) $\tau = \pm\sqrt{2}(u^4+1)^{-\frac{1}{2}}$.

6.13.15 Prove that the Gaussian curvature K of the surface, the asymptotic lines of which intersect at a constant angle, is proportional to the square of the average curvature H.

HINT:

Let the given surface have as its first fundamental form $(ds)^2 = g_{ik}du^i du^k$. Using **5.7** and **6.7.5**, show that the angle θ between the asymptotic lines on the surface is given by

$$\tan^2\theta = -K/H^2 .$$

6.13.16 Determine the torsion $\bar{\tau}$ of the asymptotic lines on the binormal surface of $\boldsymbol{r} = \boldsymbol{r}(s)$.

ANSWER:

$\bar{\tau} = \dfrac{\tau}{u^2\tau^2 + 1}$, where u is the distance of a point P along the binormal and τ is the torsion of the given curve.

6.13.17 Find the asymptotic lines on the surface of revolution $\boldsymbol{r} = \{u\cos v,\ u\sin v,\ f(u)\}$. In particular, consider the case $f(u) = \pm u^{\frac{1}{2}}$.

ANSWER:

$$v = \pm\int\left[-\frac{f''(u)}{uf'(u)}\right]^{\frac{1}{2}} du .$$

In the special case, $u = \pm k\exp(v\sqrt{2})$.

6.13.18 Find the asymptotic lines on the surface $\boldsymbol{r} = \{g(u)\cos v,\ g(u)\sin v,\ u\}$. In particular, consider the case $g(u) = k\cosh(u/k)$.

ANSWER:

$v = \pm\int[g''(u)\{g(u)\}^{-1}]^{\frac{1}{2}} du$.

In the special case, $u = \pm kv + \text{const}$.

6.14 LINES OF CURVATURE

6.14.1 Show that $g_{12} = h_{12} = 0$ is a necessary and sufficient condition for the coordinate curves to be lines of curvature.

6.14.2 Reconsider **6.11.11**. Under what conditions is a system of

curves on S^* orthogonal, if the corresponding system of curves on S is orthogonal.

HINT:

Assume that the coordinate curves on S form an orthogonal net and use (*) in **6.11.11**.

6.14.3 Find the equation of the lines of curvature on the surface defined by $z=f(x, y)$.

6.14.4 Find the lines of curvature on the surface

$$z = \arctan \frac{y}{x}$$

ANSWER:

$\log(u+\{1+u^2\}^{\frac{1}{2}}) = \pm(v+k)$, $u^2=x^2+y^2$, $v=y/x$.

6.14.5 Find the lines of curvature on the paraboloid

$$z = ax^2 + by^2 .$$

ANSWER:

The vector representation of the paraboloid is

$$\boldsymbol{r} = \{x, y, ax^2 + by^2\} .$$

Hence, the differential equation defining the lines of curvature is

$$4a^2bxy(\mathrm{d}x)^2 + (a-b+4ab^2y^2-4a^2bx^2)\mathrm{d}x\,\mathrm{d}y - 4ab^2xy(\mathrm{d}y)^2 = 0 . \quad (1)$$

Multiplying (1) by xy and introducing the transformation

$$u = ax^2 , \; v = by^2,$$

equation (1) becomes

$$(u\dot{v}-v)(\dot{v}+1) - \frac{a-b}{4ab}\dot{v} = 0 . \quad (2)$$

If $a-b\neq 0$, (2) reduces to the Clairaut differential equation

$$v - u\dot{v} = \frac{(b-a)\dot{v}}{4ab(\dot{v}+1)} .$$

Hence, the solution is

$$aCx^2 - by^2 = \frac{(a-b)C}{4ab(C+1)}, \text{ where } C = \text{const.}$$

When $a=b$ the lines of curvature are $y=C_1x$, $x^2+y^2=C_2$.

6.15 GEODESIC CURVATURE AND GEODESICS

6.15.1 Determine the geodesic curvature κ_g of the lines of longitude and latitude on a surface of revolution.

ANSWER:

$\kappa_g = 0$, $\{u(1+\{f'(u)\}^2)^{\frac{1}{2}}\}^{-1}$.

6.15.2 At the point $A=(a, 0, 0)$, find the geodesic curvature of the curve of section between the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the plane

$$\frac{x}{a} + \frac{y}{b} = 1.$$

ANSWER:

It follows from **5.9** and Note 2 in **5.15** that

$$\kappa_g = \kappa_n \tan\theta.$$

In this problem, θ is the angle between the normal plane to the ellipsoid at A (which is parallel to the x, y-plane) and the given plane. Hence, $\tan\theta = b/a$. The required normal section at A is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0.$$

Hence, since the normal curvature is $\kappa_n = a/c^2$, one has $\kappa_g = b/c^2$.

6.15.3 At the point $C=(0, 0, c)$ determine the geodesic curvature of the curve of intersection of the ellipsoid in **6.15.2** with the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\,.$$

HINT:

Use Euler's formula (see **5.10**) to obtain $\kappa_n=\dfrac{2c}{a^2+b^2}$. Proceeding as in **6.15.2**, one obtains

$$\kappa_g=2ab(a^2+b^2)^{-\frac{3}{2}}\,.$$

6.15.4 Find the geodesics on the cone $x^2+y^2-z^2=0$.

ANSWER:

The equation of the cone has the form

$$\boldsymbol{r}=\{u\sin(\pi/4)\cos v,\quad u\sin(\pi/4)\sin v,\quad u\cos(\pi/4)\}\,.$$

It suffices to consider the equation

$$2\frac{\mathrm{d}}{\mathrm{d}s}(g_{2i}\dot{u}^i)=\frac{\partial g_{ik}}{\partial u^2}\dot{u}^j\dot{u}^k\qquad(u^1=u,\ u^2=v)\,.$$

Elimination of $\mathrm{d}s$ between this equation and $(\mathrm{d}s)^2=g_{ik}\mathrm{d}u^i\mathrm{d}u^k$ gives $u^4\mathrm{d}v^2=h^2[2\mathrm{d}u^2+u^2\mathrm{d}v^2]$. This last equation solves to give

$$\frac{h}{u}=\sin\left(\frac{c\pm v}{\sqrt{2}}\right).$$

6.15.5 Show that the geodesics on cylinders are helices.

6.15.6 Find the geodesics on a general surface of revolution.

ANSWER:

$$v=c\pm h\int\left(\frac{1+(f')^2}{u^2-h^2}\right)^{\frac{1}{2}}\frac{\mathrm{d}u}{u}\,,$$

where c and h are constants and $f'=\mathrm{d}f(u)/\mathrm{d}u$.

6.15.7 Find the geodesics on the surface

$$(x^2+y^2)^{\frac{1}{2}}=c\cosh\frac{z}{c}\,.$$

ANSWER:

$$v = c_2 \pm c_1 \int [(u^2 - c^2)(u^2 - c_1^2)]^{-\frac{1}{2}} \, du \,,$$

where c_1, c_2 are constants.

6.15.8 Find the equation of the geodesics on the surface $z = f(x, y)$.

ANSWER:

$$\frac{d^2y}{dx^2}(1 + p^2 + q^2) = \left\{ r + 2s\frac{dy}{dx} + t\left(\frac{dy}{dx}\right)^2 \right\} \left\{ p\frac{dy}{dx} - q \right\},$$

where p, q, r, s, t are the abbreviations of Monge for the derivatives of z; or

$$\frac{d^2y}{dx^2} - (\Gamma^1_{11} - 2\Gamma^2_{12})\frac{dy}{dx} + (\Gamma^2_{22} - 2\Gamma^1_{12})\left(\frac{dy}{dx}\right)^2 - \Gamma^1_{22}\left(\frac{dy}{dx}\right)^3 + \Gamma^2_{11} = 0 \,,$$

where Γ^i_{jk} are the Christoffel symbols defined in **5.12**.

6.15.9 Find the geodesics on the surface $z = \alpha x + \phi(y)$, where α is a constant and $\phi(y)$ is some function of y.

ANSWER:

The equation defining the geodesic is

$$\{1 + \alpha^2 + (\phi')^2\} u \frac{du}{dy} = \alpha\phi'' u^3 - \phi'\phi'' u^2 \,,$$

where $u = dy/dx$ and the dash denotes differentiation with respect to y. The trivial solution gives $y =$ constant. The non-trivial solution is the solution of the equation of Bernoulli

$$\frac{du}{dy} + \frac{\phi'\phi''}{F(y)} u = \frac{\alpha\phi''}{F(y)} u^2 \qquad F(y) = 1 + \alpha^2 + (\phi')^2 \,.$$

Using the standard method for solving Bernoulli's equation, the solution is found to be

$$x + c_2 = \int \Phi(y)[F(y)]^{\frac{1}{2}} \, dy,$$

where c_2 is a constant and

$$\Phi(y) = c_1 - \alpha \int \phi''(y)[F(y)]^{-\frac{3}{4}} \mathrm{d}y\,,$$

where c_1 is a constant.

6.15.10 Show that the geodesics on the surface $S(x, y, z) = 0$ are defined by the equation

$$\begin{vmatrix} S_x & \mathrm{d}x & \mathrm{d}^2x \\ S_y & \mathrm{d}y & \mathrm{d}^2y \\ S_z & \mathrm{d}z & \mathrm{d}^2z \end{vmatrix} = 0 \qquad (S_x\mathrm{d}x + S_y\mathrm{d}y + S_z\mathrm{d}z = 0)\,.$$

6.16 THE MAPPING OF SURFACES

6.16.1 Show that there exists a conformal mapping of the surface $x^2 + y^2 = 2z$ onto a region of the x, y-plane.

ANSWER:

Since the given surface has the form $\boldsymbol{r} = \{u\cos v,\ u\sin v,\ u^2/2\}$, it is seen that

$$(\mathrm{d}s)^2 = u^2\left[\frac{1+u^2}{u^2}\,\mathrm{d}u^2 + \mathrm{d}v^2\right].$$

Introducing the transformation

$$\left(\frac{1+u^2}{u^2}\right)^{\frac{1}{2}} \mathrm{d}u = \mathrm{d}r, \quad \mathrm{d}v = \mathrm{d}\theta\,, \tag{1}$$

the equation for $\mathrm{d}s$ becomes

$$(\mathrm{d}s)^2 = u^2[(\mathrm{d}r)^2 + (\mathrm{d}\theta)^2]\,. \tag{2}$$

Since the coefficients of (2) are proportional to those of $(\mathrm{d}x)^2 + (\mathrm{d}y)^2$, the existence of a conformal mapping is proved (see **5.16**). The mapping $(u, v) \to (r, \theta)$, defined by (1), is the conformal mapping.

6.16.2 Show that the torus can be mapped conformally onto a region of the x, y-plane.

HINT:

Taking the equation of the torus in the form

$\boldsymbol{r} = \{(R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta\}$, where $0 \leqslant \phi < 2\pi$, $-\pi \leqslant \theta < \pi$, and R and r are constants such that $R > r$, proceed as in **6.16.1**.

6.16.3 Of what type is the mapping of the surface S onto S' such that the coordinate curves on S, which form an isometric net, are mapped onto an isometric net on S'.

6.16.4 Show that the mapping of the helicoid S onto the unit sphere by the normals to S is conformal.

ANSWER:

Since S is given by $\boldsymbol{r} = \{u\cos v,\ u\sin v,\ kv\}$,

$$(\mathrm{d}s)^2 = (\mathrm{d}u)^2 + (k^2 + u^2)(\mathrm{d}v)^2 .$$

The unit normal on S is

$$\boldsymbol{N} = \{k^2 + u^2\}^{-\frac{1}{2}}\{k\sin v,\ -k\cos v, u\} .$$

As a position vector, $\boldsymbol{N}$ defines the unit sphere onto which S is mapped by $\boldsymbol{N}$. The line element $\mathrm{d}\bar{s}$ for this sphere is defined by

$$(\mathrm{d}\bar{s})^2 = \frac{k^2}{(k^2 + u^2)^2}\{(\mathrm{d}u)^2 + (k^2 + u^2)(\mathrm{d}v)^2\} .$$

Hence the mapping is conformal.

6.16.5 Show that the mapping of the surface S

$$\boldsymbol{r} = \left\{ u\cos v,\ u\sin v,\ k\log\left[\frac{u + (u^2 - k^2)^{\frac{1}{2}}}{k}\right]\right\}$$

onto the unit sphere by the normals to S is conformal.

HINT:

Proceed as in **6.16.4**.

6.16.6 The *loxodromes* of a surface are the curves which cut the lines of longitude of the surface at a constant angle.

Let the given surface be a sphere of radius k.

(a) Find the equation of the loxodromes.

(b) Consider the stereographic projection of the sphere from the north pole onto the equatorial plane. Determine the curves into which the loxodromes are mapped by this projection.

(c) Show that the stereographic projection of the sphere from the north pole onto the equatorial plane defines a conformal mapping.

ANSWER:

(a) $v \pm \tan \alpha \log\left\{\tan\left(\frac{u}{2}+\frac{\pi}{4}\right)\right\} = \text{const.}$, where the curves $u = \text{const.}$ and $v = \text{const.}$ define the lines of latitude and longitude.

(b) The stereographic projection $(u, v) \to (r, \theta)$, where r and θ are polar coordinates, is defined by

$$r = k \tan\left(\frac{u}{2}+\frac{\pi}{4}\right), \quad \theta = v .$$

Hence, the stereographic projections of the loxodromes have the form

$$r = e^{c_1\theta + c_2} ,$$

which are logarithmic spirals.

6.16.7 The tractrix is a plane curve for which the segment of the tangent between the point of contact of the tangent and the point of intersection of the tangent with a fixed line p has the constant length k. The rotation of the tractrix about the direction p describes a pseudosphere.

On the pseudosphere introduce the curvilinear coordinates (u, v) where u is the arc length of the meridians (lines of longitude-tractrices) measured from their singular points, and v is the angle between a plane through an arbitrary meridian and a plane through a fixed meridian.

(a) Determine the first fundamental form for the pseudosphere in terms of u and v.

(b) Determine the type of mapping which maps the pseudosphere onto the x^*, y^*-Euclidean plane according to $x^* = v$, $y^* = e^u$.

(c) Find the geodesics of the pseudosphere.

(d) Find the loxodromes of the pseudosphere.

ANSWER:

(a) Let the tractrix lie in the x, z-plane, the direction p be the z-axis, and the length k be 1, then the two branches of the tractrix are defined by the differential equation

$$\frac{dx}{dz} = \pm x\,(1-x^2)^{-\frac{1}{2}}\,.$$

Consider the branch defined by the negative sign in (1).

The meridians $v = \text{const.}$ and the lines of latitude $u = \text{const.}$ interscet orthogonally, therefore

$$(ds)^2 = g_{11}(du)^2 + g_{22}(dv)^2\,.$$

The line element on the meridian is $g_{11}^{\frac{1}{2}}\,du$ while the line element on the lines of latitude is $g_{22}^{\frac{1}{2}}\,dv$. Using the definitions of u and v above, it is seen that

$$g_{11} = 1, \quad g_{22} = x^2\,.$$

Since $(du)^2 = (dx)^2 + (dz)^2$, it is seen from (1) that $x = e^u$. Hence,

$$(ds)^2 = (du)^2 + e^{-2u}(dv)^2\,.$$

(b) The mapping is condormal.

(c) Introducing the transformation $\bar{x} = v$, $\bar{y} = e^u$, it is seen from (2) that

$$(ds)^2 = (\bar{y})^{-2}[(d\bar{x})^2 + (d\bar{y})^2]\;;$$

i.e., that $\bar{g}_{11} = \bar{g}_{22} = (\bar{y})^{-2}$, $\bar{g}_{12} = 0$. Using (5.15.1) the equation for the geodesics is found to be

$$\bar{y}\bar{y}'' + (\bar{y}')^2 + 1 = 0\,.$$

This latter equation yields

$$(\bar{x}-k)^2 + (\bar{y})^2 = b^2\,.$$

(d) The loxodromes are the curves

$$e^u = k_1 v + k_2,\ u \geqslant 0, \qquad k_1 \text{ and } k_2 = \text{const.}$$

6.16.8 On the unit sphere let the coordinate curves be the lines of

latitude $u=$const. and the lines of longitude $v=$const. On the x, y-plane let the coordinate curves be defined by

(a) $x=\rho(u)\cos v,\ y=\rho(u)\sin v,$

(b) $x=\phi(u, v),\ y=u,$

(c) $x=v,\ y=\psi(u, v),$

(d) $x=f(v),\ y=f(v)\cdot h(u).$

Between the points M on the sphere and the points $\bar{M}$ on the plane the mappings α_1, α_2, α_3 and α_4 are defined by (a), (b), (c) and (d), respectively, such that corresponding points M and $\bar{M}$ have the same values for u and v. Determine the form of the functions ρ, ϕ, ψ, f and h so that the mappings α_1, α_2, α_3, and α_4 are area preserving. In (a), the function u is chosen so that the south pole is mapped onto a point. For the other cases the required functions are taken in their simplest forms. Describe the curves into which the mappings α_1, α_2, α_3 and α_4 map the lines of latitude and longitude.

ANSWER:

(a) $\rho(u)=(2\sin u+k)^{\frac{1}{2}}$. Since the south pole is determined by $u=-\pi/2$ and since α_1 will map the south pole onto a point only if $\rho(-\pi/2)=0$, it follows that $C=2$. Hence

$$\rho(u)=2\cos\left(\frac{\pi}{4}-\frac{u}{2}\right).$$

The lines of latitude are mapped onto concentric circle with center at the origin and radius between 0 and 2. The lines of longitude are the radii of the largest circle.

(b) $\phi(u, v)=v\cos u+\chi(u)$, where $\chi(u)$ is an arbitrary function of u. The simplest form is $\phi(u, v)=v\cos u$. The lines of latitude are mapped into the lines $y=$const., $-\pi/2\leqslant y\leqslant\pi/2$, and the lines of longitude are the curves $x=C\sin y$, $-\pi\leqslant C\leqslant\pi$.

(c) $\psi(u, v)=\sin u+\psi_1(v)$, where $\psi_1(v)$ is an arbitrary function. The sphere is mapped onto a rectangular grid with center at the origin. The lines of latitude and longitude are represented by the lines $y=C_1$, $-1\leqslant C_1\leqslant 1$, and $x=C_2$, $-\pi\leqslant C_2\leqslant\pi$, respectively.

(d) The condition to be satisfied is

$$f(v)\frac{\mathrm{d}f(v)}{\mathrm{d}v}\cdot\frac{\mathrm{d}h(u)}{\mathrm{d}u}=\cos u\,.$$

The simplest functions satisfying this condition are

$$f(v)=v^{\frac{1}{2}}, \quad h(u)=2\sin u\,.$$

The lines of latitude and longitude are represented by the lines $x=C_1$, $-\pi\leqslant C_1\leqslant\pi$, and $y=C_2x$, $-2\leqslant C_2\leqslant 2$.

6.16.9 Consider the tangent surface of the helix

$$\boldsymbol{r}=\{c\cos\phi,\ c\sin\phi,\ k\phi\}\,. \tag{1}$$

(a) Show that the isometric mapping of this surface onto the x,y-plane maps the helices on this surface onto concentric circles.

(b) What condition must the constants c and k satisfy in order that the lengths of the helices between $\phi=2m\pi$ and $\phi=2(m+n)\pi$ be mapped onto the circumferences of the corresponding circles?

ANSWER:

(a) The length of the given helix measured from $\phi=0$ to $\phi=\phi$ is $u=k_0\phi$, where $k_0=(c^2+k^2)^{\frac{1}{2}}$; and hence, (1) has the form

$$\bar{\boldsymbol{r}}(u)=\{a\cos\frac{u}{k_0},\ a\sin\frac{u}{k_0},\ \frac{k}{k_0}u\}\,.$$

The tangent surface is then defined by

$$\boldsymbol{r}(u,\theta)=\boldsymbol{r}(u)+v\bar{\boldsymbol{r}}'(u), \quad \bar{\boldsymbol{r}}'=\frac{\mathrm{d}\bar{\boldsymbol{r}}}{\mathrm{d}u}\,.$$

It follows that on the tangent surface

$$(\mathrm{d}\bar{s})^2=\left(1+\frac{c^2}{k_0^4}v^2\right)(\mathrm{d}u)^2+2\,\mathrm{d}u\,\mathrm{d}v+(\mathrm{d}v)^2\,.$$

For the mapping $\alpha\colon(u,v)\to(x,y)$ defined by

$$x=r\cos\frac{u}{r}-v\sin\frac{u}{r}, \quad y=r\sin\frac{u}{r}+v\cos\frac{u}{r},$$

where r is a constant,

$$(\mathrm{d}s)^2=\left(1+\frac{v^2}{r^2}\right)(\mathrm{d}u)^2+2\mathrm{d}u\,\mathrm{d}v+(\mathrm{d}v)^2\,.$$

Hence, if $r=(c^2+k^2)c^{-1}$, then this mapping is isometric and the curves $v=$ constant (helices) are mapped by α onto the circles

$$x^2+y^2=r^2+v^2\,.$$

(b) $k=\pm c(n^2-1)^{\frac{1}{2}}$.

6.16.10 Show that there corresponds to a surface

$$r=\{u\cos v,\ u\sin v,\ c\log[(u+(u^2-c^2)^{\frac{1}{2}})c^{-1}]\}\,,$$

a helicoid, onto which this surface is mapped isometrically.

ANSWER:

Let the helicoid be

$$\bar{r}=\{r\cos\theta,\ r\sin\theta,\ k\theta\},\qquad k=\text{const.}$$

The required helicoid is defined by

$$r=(u^2-c^2)^{\frac{1}{2}},\qquad \theta=v\,,$$

where $c=\pm k$.

6.16.11 Find the curves C such that there exists a generalised area preserving map between the binormal surface of C and a helicoid. In the mapping generators of the binormal surface are to correspond to the generators of the helicoid. When is the mapping isometric?

ANSWER:

The curves C are all the curves with constant torsion τ. The mapping is isometric when the pitch of he helicoid equals unity.

6.16.12 Consider the right circular cone

$$r=\{u\sin\beta\cos v,\ u\sin\beta\sin v,\ u\cos\beta\}\,.$$

Let ω be the plane passing through the vertix V of the cone and perpendicular to its axis. Corresponding to a point M of the cone a point $\overline{M}$ is defined on the plane ω by the mapping $\alpha\colon M\to\overline{M}$, where M is the point of intersection between the plane ω and the circle with radius VM and lying in the plane containing VM and the axis of the cone. What type of mapping is defined by α?

ANSWER:

α is a generalised area preserving mapping.

6.16.13 On the sphere with radius k a fixed point P is given. Let M be an arbitrary point on the sphere, and N be the mid-point of the shortest line on the sphere which connects P and M. Let α be the mapping $M \to \overline{M}$, where $\overline{M}$ is the orthogonal projection of N onto the tangent plane to the sphere at P.

(a) What type of mapping is defined by α?

(b) What type of mapping is defined by β: $M \to M^*$, where $PM^* = 2PM$?

ANSWER:

(a) α is a generalised area preserving mapping.

(b) β is an area preserving mapping.

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